

# Appendix A

## Optical polarization tutorial

### A.1 Polarization of light

Consider a classical plane electromagnetic wave propagating along the (horizontal)  $z$ -axis. The electric field vector is given by

$$\vec{E} = \text{Re}[(\alpha\vec{e}_H + \beta\vec{e}_V)e^{ikz-i\omega t}], \quad (\text{A.1})$$

where  $\vec{e}_H$  and  $\vec{e}_V$  are unit horizontal and vertical vectors, respectively;  $\alpha$  and  $\beta$  are the complex amplitudes. The intensity of light in each polarization is proportional to:

$$I_H = |\alpha|^2; \quad (\text{A.2a})$$

$$I_V = |\beta|^2. \quad (\text{A.2b})$$

The total intensity is  $I_{\text{total}} = |\alpha|^2 + |\beta|^2$ .

**Definition A.1** According to the trajectory of the tip of the electric field vector at a specific point in the phase space (see Ex. A.1 below), the polarization patterns of plane waves (A.1) are classified as follows:

- *Linear polarization*:  $\arg \beta = \arg \alpha (\pm\pi)$ . There are important special cases:
  - *Horizontal polarization*:  $\beta = 0$
  - *Vertical polarization*:  $\alpha = 0$
- *Circular polarization*:  $\begin{cases} \arg \beta = \arg \alpha \pm \frac{\pi}{2} \\ |\alpha| = |\beta| \end{cases}$
- *Elliptical polarization*: all other cases.

**Exercise A.1** Suppose we measure the electric field at a particular point in space. Show that

- a) when the polarization is linear, the tip of the electric field vector oscillates along a straight line with the amplitude  $\sqrt{\alpha^2 + \beta^2}$ ;
- b) when the polarization is circular, the tip of the electric field vector follows a circular pattern of radius  $\alpha = \beta$ ;
- c)\* when the polarization is elliptical, the tip of the electric field vector follows an elliptical pattern. Find the major and minor semiaxes of the ellipse as well as the angle between the axes and the horizontal.

Suppose we can isolate a single photon from a polarized wave. This photon's polarization is a simple physical system: the polarization state of the photon is

$$|\psi\rangle = \alpha |H\rangle + \beta |V\rangle, \quad (\text{A.3})$$

where  $|H\rangle$  and  $|V\rangle$  are the horizontal and vertical polarization states which make up an orthonormal basis. Because physical states have to be normalized, we have to rescale  $\alpha$  and  $\beta$  so that  $\alpha^2 + \beta^2 = 1$ .

## A.2 Polarizing beam splitter

The *polarizing beam splitter* (Fig. A.2) is an important optical device for the analysis of polarization. It is a transparent cube consisting of two triangular prisms glued to each other, constructed to transmit horizontally polarized light, but reflect vertically polarized. If a classical wave (A.1) is incident on such a beam splitter, the intensity of the transmitted and reflected light will be proportional to  $\alpha^2$  and  $\beta^2$ , respectively.

Let us assume a single photon (A.3) hits a polarizing beam splitter. The photon is the smallest energy portion of light, and cannot be divided into parts. Will it be transmitted or reflected? The experiment shows that the outcome will be random: the photon will go through the beam splitter with a probability  $\alpha^2$ , and be reflected with a probability  $\beta^2$ . Such probabilistic behavior of quantum objects is very typical; we generalize it as the Second Quantum Mechanics Postulate.

If a large number  $N$  of photons are incident on the polarizing beam splitter (e.g. in the case of a classical wave), on average  $\alpha^2 N$  of them will be transmitted, and  $\beta^2 N$  reflected. This is equivalent to the intensities being proportional to  $\alpha^2$  and  $\beta^2$ , respectively, so the quantum theory is consistent with the classical.

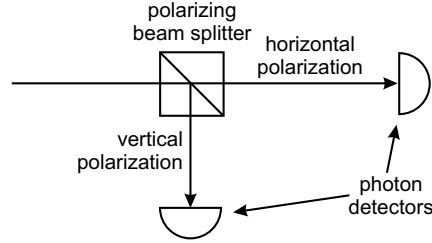


Figure A.1: Polarizing beam splitter.

## A.3 Waveplates

Another important optical instrument is the *waveplate*. It is a birefringent<sup>1</sup> optical crystal which introduces a phase delay  $\Delta\phi$  between the ordinary and extraordinary field components transmitted through it. Two kinds of waveplates are manufactured commercially:  $\lambda/2$ -plate with  $\Delta\phi = \pi$  and  $\lambda/4$ -plate with  $\Delta\phi = \pi/2$ .

Let us calculate transformations that a photon will undergo by propagating through a waveplate with its axis oriented vertically.

**Half-wave plate:**

$$\alpha |H\rangle + \beta |V\rangle \longrightarrow \alpha |H\rangle + \beta |V\rangle e^{i\pi} = \alpha |H\rangle - \beta |V\rangle.$$

For the specific case of linear polarization (at angle  $\theta$  to horizontal):

$$\cos\theta |H\rangle + \sin\theta |V\rangle \longrightarrow \cos\theta |H\rangle - \sin\theta |V\rangle = \cos\theta |H\rangle + \sin(-\theta) |V\rangle.$$

The wave remains linearly polarized, but its polarization angle switches to  $-\theta$ . Note that waves polarized at angles  $-\theta$  and  $\pi - \theta$  are equivalent, so we can say that under the action of a half-wave plate the linear polarization vector makes a “mirror flip” around the optical axis.

<sup>1</sup>*Birefringence*, or *double refraction*, is an optical property of certain materials, primarily crystals. Birefringent materials have anisotropic structure, such that a light wave polarized along a particular direction (called *optical axis*) has index of refraction which is different from waves polarized in perpendicular directions. These waves are called *extraordinary* and *ordinary*, respectively. Polarization properties of a wave that is neither ordinary nor extraordinary evolve as the wave propagates through the material.

This conclusion can be generalized: when a wave linearly polarized at angle  $\theta$  propagates through a half-wave plate with its optical axis oriented at angle  $\varphi$ , the polarization will rotate by  $2|\varphi - \theta|$  (Fig. A.3).

**Exercise A.2** What will a half-wave plate do with the elliptical polarization? Make a calculation and give a graphical description.

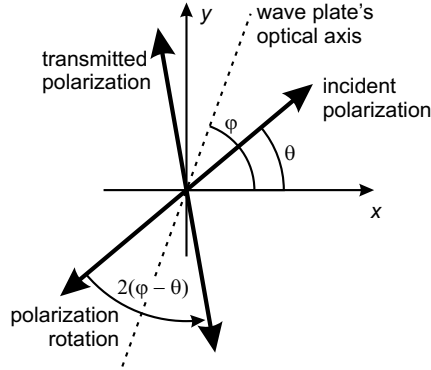


Figure A.2: Polarization rotation by a  $\lambda/2$  plate.

**Quarter-wave plate:**

$$\alpha |H\rangle + \beta |V\rangle \quad (\text{general case}) \quad \longrightarrow \quad \alpha |H\rangle + e^{i\frac{\pi}{2}} \beta |V\rangle = \alpha |H\rangle + i\beta |V\rangle \quad (\text{A.4})$$

$$\frac{|H\rangle + |V\rangle}{\sqrt{2}} \quad (+45^\circ \text{ polarization}) \quad \longrightarrow \quad \frac{|H\rangle + i|V\rangle}{\sqrt{2}} \quad (\text{right circular}) \quad (\text{A.5})$$

$$\frac{|H\rangle - |V\rangle}{\sqrt{2}} \quad (-45^\circ \text{ polarization}) \quad \longrightarrow \quad \frac{|H\rangle - i|V\rangle}{\sqrt{2}} \quad (\text{left circular}) \quad (\text{A.6})$$

$$\frac{|H\rangle + i|V\rangle}{\sqrt{2}} \quad (\text{right circular}) \quad \longrightarrow \quad \frac{|H\rangle + i^2|V\rangle}{\sqrt{2}} = \frac{|H\rangle - |V\rangle}{\sqrt{2}} \quad (-45^\circ \text{ polarization}) \quad (\text{A.7})$$

$$\frac{|H\rangle - i|V\rangle}{\sqrt{2}} \quad (\text{left circular}) \quad \longrightarrow \quad \frac{|H\rangle - i^2|V\rangle}{\sqrt{2}} = \frac{|H\rangle + |V\rangle}{\sqrt{2}} \quad (+45^\circ \text{ polarization}) \quad (\text{A.8})$$

$$(\text{A.9})$$

We see that a  $\lambda/4$ -plate with the optical axis oriented horizontally (or vertically) converts between the circular and  $\pm 45^\circ$  polarizations.