

Appendix A

Optical polarization tutorial

A.1 Polarization of light

Consider a classical plane electromagnetic wave propagating along the (horizontal) z -axis. The electric field vector is given in the complex form by

$$\vec{E} = \text{Re}[(A_H e^{i\varphi_H} \vec{e}_H + A_V e^{i\varphi_V} \vec{e}_V) e^{ikz - i\omega t}], \quad (\text{A.1})$$

where \vec{e}_H and \vec{e}_V are unit horizontal and vertical vectors, respectively; A_H and A_V are the real amplitudes and the φ 's are the real phases associated with the vertical and horizontal components of the wave. This wave can be written in terms of real numbers as

$$\vec{E} = A_H \vec{e}_H \cos(kz - \omega t + \varphi_H) + A_V \vec{e}_V \cos(kz - \omega t + \varphi_V), \quad (\text{A.2})$$

The intensity of light in each polarization is proportional to:

$$I_H \propto A_H^2; \quad (\text{A.3a})$$

$$I_V \propto A_V^2. \quad (\text{A.3b})$$

The total intensity is $I_{\text{total}} \propto |\alpha|^2 + |\beta|^2$.

Definition A.1 According to the trajectory of the tip of the electric field vector at a specific point in the phase space (see Ex. A.1 below), the polarization patterns of plane waves (A.1) are classified as follows:

- *Linear polarization*: $\varphi_H = \varphi_V (\pm\pi)$. There are important special cases:
 - *Horizontal polarization*: $A_V = 0$
 - *Vertical polarization*: $A_H = 0$
- *Circular polarization*: $\begin{cases} \varphi_H = \varphi_V \pm \frac{\pi}{2} \\ A_H = A_V \end{cases}$
- *Elliptical polarization*: all other cases.

Exercise A.1 Suppose we measure the electric field at a particular point in space. Show that

- a) when the polarization is linear, the tip of the electric field vector oscillates along a straight line with the amplitude $\sqrt{A_H^2 + A_V^2}$;
- b) when the polarization is circular, the tip of the electric field vector follows a circular pattern of radius $A_H = A_V$;
- c)* when the polarization is elliptical, the tip of the electric field vector follows an elliptical pattern. Find the major and minor semi-axes of the ellipse as well as the angle between the axes and the horizontal.

A.2 Polarizing beam splitter

The *polarizing beam splitter (PBS)* (Fig. A.2) is an important optical device for the analysis of polarization. It is a transparent cube consisting of two triangular prisms glued to each other, constructed to transmit horizontally polarized light, but reflect vertically polarized. If a classical wave (A.1) is incident on such a beam splitter, the intensity of the transmitted and reflected light will be proportional to A_H^2 and A_V^2 , respectively.

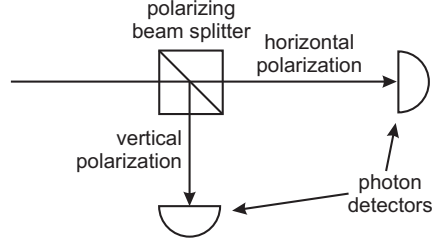


Figure A.1: Polarizing beam splitter.

A.3 Waveplates

Another important optical instrument is the *waveplate*. It is a birefringent¹ optical crystal which introduces a phase delay $\Delta\phi$ between the ordinary and extraordinary field components transmitted through it. Two kinds of waveplates are manufactured commercially: $\lambda/2$ -plate with $\Delta\phi = \pi$ and $\lambda/4$ -plate with $\Delta\phi = \pi/2$.

Let us calculate transformations that a polarized field will undergo by propagating through a waveplate with its axis oriented vertically. To simplify the notation, we will write the polarization in terms of photon states. The transformation of classical fields under the action of waveplates follows the same rules.

Half-wave plate

$$\alpha |H\rangle + \beta |V\rangle \longrightarrow \alpha |H\rangle + \beta |V\rangle e^{i\pi} = \alpha |H\rangle - \beta |V\rangle.$$

For the specific case of linear polarization (at angle θ to horizontal):

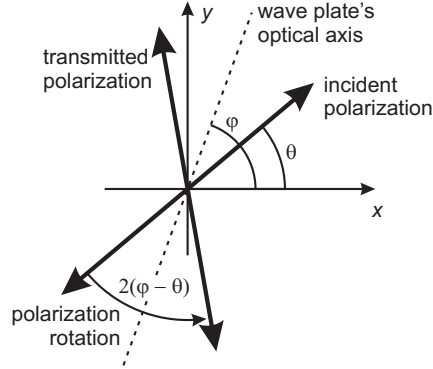
$$\cos\theta |H\rangle + \sin\theta |V\rangle \longrightarrow \cos\theta |H\rangle - \sin\theta |V\rangle = \cos\theta |H\rangle + \sin(-\theta) |V\rangle.$$

The wave remains linearly polarized, but its polarization angle switches to $-\theta$. Note that waves polarized at angles $-\theta$ and $\pi - \theta$ are equivalent, so we can say that under the action of a half-wave plate the linear polarization vector makes a “mirror flip” around the optical axis.

This conclusion can be generalized: when a wave linearly polarized at angle θ propagates through a half-wave plate with its optical axis oriented at angle φ , the polarization will rotate by $2|\varphi - \theta|$ (Fig. A.3).

Exercise A.2 What will a half-wave plate do with the elliptical polarization? Make a calculation and give a graphical description.

¹*Birefringence*, or *double refraction*, is an optical property of certain materials, primarily crystals. Birefringent materials have anisotropic structure, such that a light wave polarized along a particular direction (called *optical axis*) has index of refraction which is different from waves polarized in perpendicular directions. These waves are called *extraordinary* and *ordinary*, respectively. If a wave that is neither ordinary nor extraordinary, its polarization state will change as the wave propagates through the material.

Figure A.2: Polarization rotation by a $\lambda/2$ plate.**Quarter-wave plate**

The general formula for the transformation under the action of a quarter-wave plate is similar to that due to a half-wave plate except that the phase delay is halved:

$$\alpha |H\rangle + \beta |V\rangle \longrightarrow \alpha |H\rangle + e^{i\frac{\pi}{2}} \beta |V\rangle = \alpha |H\rangle + i\beta |V\rangle \quad (\text{A.4})$$

The following special cases are particularly important.

$$\frac{|H\rangle + |V\rangle}{\sqrt{2}} \quad (+45^\circ \text{ polarization}) \longrightarrow \frac{|H\rangle + i|V\rangle}{\sqrt{2}} \quad (\text{right circular}) \quad (\text{A.5})$$

$$\frac{|H\rangle - |V\rangle}{\sqrt{2}} \quad (-45^\circ \text{ polarization}) \longrightarrow \frac{|H\rangle - i|V\rangle}{\sqrt{2}} \quad (\text{left circular}) \quad (\text{A.6})$$

$$\frac{|H\rangle + i|V\rangle}{\sqrt{2}} \quad (\text{right circular}) \longrightarrow \frac{|H\rangle + i^2|V\rangle}{\sqrt{2}} = \frac{|H\rangle - |V\rangle}{\sqrt{2}} \quad (-45^\circ \text{ polarization}) \quad (\text{A.7})$$

$$\frac{|H\rangle - i|V\rangle}{\sqrt{2}} \quad (\text{left circular}) \longrightarrow \frac{|H\rangle - i^2|V\rangle}{\sqrt{2}} = \frac{|H\rangle + |V\rangle}{\sqrt{2}} \quad (+45^\circ \text{ polarization}) \quad (\text{A.8})$$

We see that a $\lambda/4$ -plate with the optical axis oriented horizontally (or vertically) interconverts between the circular and $\pm 45^\circ$ polarizations. Note that, similarly to the definition of the circular polarization state (see Note 1.6), the action of the quarter wave plate is not uniquely defined. This is because the extraordinary index of refraction can be either higher or lower than the ordinary, and, accordingly, the ordinary and extraordinary waves can experience a relative phase delay of either $+\pi/2$ or $-\pi/2$. In this course, we will assume the quarter wave plate transformation to be as defined above.

