Appendix C

Tutorial on the Dirac delta function and the Fourier transformation

C.1 Dirac delta function

The delta function $\delta(x)$ studied in this section is a function that takes on zero values at all $x \neq 0$, and is infinite at $x = 0$, so that its integral $\int\limits_{-\infty}^{+\infty} \delta(x)dx = 1$. This function allows one to write down spatial density of a physical quantity that is concentrated in one point. For example, the density of a one-dimensional particle of mass m located at $x = a$ is written as $m\delta(x-a)$. In quantum mechanics, we use $\delta(x)$ to write, for example, the wave function of a state with a well-defined position.

The delta function belongs to the class of so-called *generalized functions*. This means that it is meaningful only as a part of an integral expression. While we may write a delta function outside of an integral, we always keep in mind that it will eventually become a part of an integral, and only then will it produce a valid result that can be used, for example, to predict an outcome of an experiment. It is not possible to provide a rigorous mathematical theory of generalized functions in the framework of this course. Below, we discuss only those properties of the delta function that are useful for physicists.

Definition C.1 The *Dirac delta function* is a generalized function such that for any function $f(x)$ that is smooth[1] and takes a value of 0 at ±∞

$$\int\limits_{-\infty}^{+\infty} \delta(x)f(x)dx = f(0) \tag{C.1}$$

Exercise C.1 Show that

a)

$$\int\limits_{-\infty}^{+\infty} \delta(x)dx = 1; \tag{C.2}$$

b) for any function $f(x)$,

$$\int\limits_{-\infty}^{+\infty} \delta(x-a)f(x)dx = f(a); \tag{C.3}$$

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[1] A *smooth* function is one that has derivatives of all finite orders.

c) for any real number a,

$$\delta(ax) = \delta(x)/|a|. \tag{C.4}$$

Exercise C.2 For the Heaviside step function $\theta(x) = \begin{cases} 0 & \text{if} x<0 \\ 1 & \text{if} x\geq 0 \end{cases}$ show that

$$\frac{d}{dx}\theta(x) = \delta(x). \tag{C.5}$$

Hint: use Eq. (C.1) with any smooth $f(x)$ vanishing at $\pm\infty$.

Note C.1 The above result can be generalized to any function $g(x)$ that has a discontinuity at the point $x = x_0$:

$$\frac{dg(x)}{dx} = \left.\frac{dg(x)}{dx}\right|_{x\neq x_0} + [g(x_0+0) - g(x_0-0)]\delta(x-x_0). \tag{C.6}$$

Definition C.2 Function

$$G_b(x) = e^{-x^2/b^2}, \tag{C.7}$$

is called the *Gaussian function.*

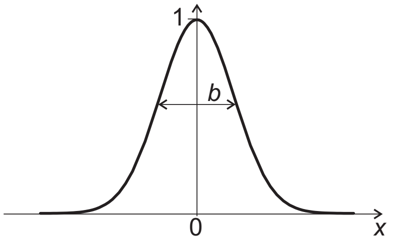

Figure C.1: Gaussian function e^{-x^2/b^2}.

Exercise C.3 Show that the integral of the Gaussian function

$$\int\limits_{-\infty}^{+\infty} e^{-x^2/b^2} dx = b\sqrt{\pi} \tag{C.8}$$

(**Hint:** use $\int\limits_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$.)

Note C.2 The delta function can be visualized as a Gaussian function $\frac{1}{b\sqrt{\pi}}G_b(x)$ of infinitely narrow width b. The factor $1/(b\sqrt{\pi})$ is chosen to make the function's integral equal to 1. We can write

$$\frac{1}{b\sqrt{\pi}}e^{-x^2/b^2} \to \delta(x) \qquad \text{for} \qquad b \to 0. \tag{C.9}$$

Exercise C.4 Show that for a smooth function $f(x)$ which takes zero values at $\pm\infty$,

$$\int\limits_{-\infty}^{+\infty} \left[\frac{d}{dx}\delta(x)\right] f(x)dx = -\left.\frac{df}{dx}\right|_{x=0}. \tag{C.10}$$

Note C.3 Because the delta function is meaningful only as a part of an integral expression, Eq. (C.10) can be rewritten as follows:

$$\left[\frac{d}{dx}\delta(x)\right] f(x) = -\delta(x)\frac{df}{dx}, \tag{C.11}$$

In other words, the expression $\frac{d}{dx}\delta(x)$ can be seen as an operator acting on functions:

$$\frac{d}{dx}\delta(x) = -\delta(x)\frac{d}{dx}. \tag{C.12}$$

C.2 Fourier transformation

This is an important integral transformation used in all branches of physics. It is used, for example, to determine the frequency spectrum of a time-dependent signal.

Definition C.3 The *Fourier transform* $\tilde{f} \equiv \mathcal{F}[f]$ of a function $f(x)$ is a function of parameter k defined as follows:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ikx} f(x) dx. \tag{C.13}$$

Note C.4 $\tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)dx$

Exercise C.5 For a real $f(x)$, $\tilde{f}(k) = \tilde{f}^*(-k)$.

Exercise C.6 Show that the Fourier transform of a Gaussian function from Ex. C.3 is a Gaussian function

$$\mathcal{F}[e^{-x^2/b^2}] = \frac{b}{\sqrt{2}} e^{-k^2 b^2/4}. \tag{C.14}$$

Exercise C.7 a) Show that in the limit $b \to 0$, Eq. (C.14) takes the form

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}. \tag{C.15}$$

b) Show that in the opposite limit, $b \to \infty$, one obtains

$$\mathcal{F}[1] = \sqrt{2\pi}\,\delta(k). \tag{C.16}$$

[**Hint:** use Eq. (C.9).]

Exercise C.8 Show that

$$\int\limits_{-\infty}^{+\infty} e^{ik_0 x} dx = 2\pi\delta(k_0) \tag{C.17}$$

Note C.5 The above equation can be straightforwardly extended:

$$\int\limits_{-\infty}^{+\infty} e^{iak_0 x} dx = 2\pi\delta(k_0)/|a|. \tag{C.18}$$

Definition C.4 The *inverse Fourier transform* $\mathcal{F}^{-1}[g]$ of a function $g(k)$ is a function of parameter x such that

$$\breve{g}(x) = \mathcal{F}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ikx} g(k) dk. \tag{C.19}$$

Exercise C.9

$$\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x). \tag{C.20}$$

Exercise C.10

$$\mathcal{F}^{-1}[f(x)] = \mathcal{F}[f(-x)]. \tag{C.21}$$

Exercise C.11 Show that, if $\tilde{f}(k) = \mathcal{F}[f(x)]$ exists, then

a)

$$\mathcal{F}[f(ax)] = \frac{1}{|a|}\tilde{f}(k/a); \tag{C.22}$$

b)

$$\mathcal{F}[f(x-a)] = e^{-ika}\tilde{f}(k); \tag{C.23}$$

c)

$$\mathcal{F}[e^{i\xi x}f(x)] = \tilde{f}(k-\xi). \tag{C.24}$$

d)

$$\mathcal{F}[df(x)/dx] = ik\tilde{f}(k). \tag{C.25}$$

Write similar rules for the inverse Fourier transformation.

Exercise C.12 Find the Fourier transform of $\delta(x+a) + \delta(x-a)$.

Definition C.5 The *convolution* of two functions is the integral

$$[f * g](x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy \tag{C.26}$$

Exercise C.13 Show that the definition of convolution is symmetric, i.e. $f * g = g * f$.

Exercise C.14 Show that any function is a convolution of itself with the delta function.

Exercise C.15 Show that, for any two functions $f(x)$ and $g(x)$,

a)

$$F[f * g] = \sqrt{2\pi}F[f] \times F[g]; \tag{C.27}$$

b)

$$F[f \times g] = \frac{1}{\sqrt{2\pi}}F[f] * F[g]. \tag{C.28}$$

Exercise C.16 Verify the above result explicitly for two Gaussian functions $G_a(x)$ and $G_b(x)$.