Appendix D

Solutions to Appendix C problems

Solution to Exercise C.1.

a) We obtain Eq. (C.2) by substituting $f(x) = 1$ into Eq. (C.1).

b) Make a replacement of the integration variable $x - a = t$. Then $\mathrm{d}t = \mathrm{d}x$ and

$$\int_{-\infty}^{+\infty} \delta(x-a) f(x) \mathrm{d}x = \int_{-\infty}^{+\infty} \delta(t) f(t+a) \mathrm{d}t \overset{(C.1)}{=} \delta(a).$$

c) Consider a function $f(x)$ and an integral

$$I = \int_{-\infty}^{+\infty} \delta(ax) f(x) \mathrm{d}x.$$

To calculate this integral, we replace the integration variable $ax = t$, so $\mathrm{d}x = \mathrm{d}t/a$. Then for positive a,

$$I = \int_{-\infty}^{+\infty} \delta(t) f(t/a) \mathrm{d}t/a = f(0)/a.$$

If a is negative, we have to exchange the integration limits:

$$I = \int_{+\infty}^{-\infty} \delta(t) f(t/a) \mathrm{d}t/a = -\int_{-\infty}^{+\infty} \delta(t) f(t/a) \mathrm{d}t/a = -f(0)/a.$$

The two equations above can be combined by writing

$$\int_{-\infty}^{+\infty} \delta(ax) f(x) \mathrm{d}x = f(0)/|a|. \tag{D.1}$$

Comparing Eqs. (C.1) and (D.1) we obtain

$$\delta(ax) = \frac{\delta(x)}{|a|}.$$

Solution to Exercise C.2. Let $\mathrm{d}\theta(x)/\mathrm{d}x = \alpha(x)$ and consider the integral

$$I = \int_{-\infty}^{+\infty} \alpha(x) f(x) \mathrm{d}x$$

for a smooth function $f(x)$ with $f(\pm\infty)=0$. Integrating by parts, we find

$$I = \theta(x)f(x)|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty} \theta(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\mathrm{d}x.$$

The first term in the above expression vanishes because $f(\pm\infty)=0$, and, according to the definition of the Heaviside function, the second term is

$$I = -\int\limits_{0}^{+\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)\mathrm{d}x = f(0).$$

Thus the generalized function $\alpha(x)$ behaves according to the definition (C.1) of the delta function, i.e. it *is* the delta function.

Solution to Exercise C.3. The result (C.8) obtains by substituting a new integration variable $x/b=t$ and using $\int\limits_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\pi}$:

$$\int\limits_{-\infty}^{+\infty} e^{-x^2/b^2}\mathrm{d}x = \int\limits_{-\infty}^{+\infty} e^{-t^2}b\,\mathrm{d}t = b\sqrt{\pi}.$$

Note that Eq. (C.8) also holds for complex b as long as $\mathrm{Re}(b)>0$.

Solution to Exercise C.4. Treating the delta function as a regular function, we integrate by parts:

$$\int\limits_{-\infty}^{+\infty}\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x)\right]f(x)\mathrm{d}x = \delta(x)f(x)|_{-\infty}^{+\infty} - \int\limits_{-\infty}^{+\infty}\delta(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]\mathrm{d}x = -\left.\frac{df}{dx}\right|_{x=0}.$$

The last equality follows from the definition of the delta function; the term $\delta(x)f(x)|_{-\infty}^{+\infty}$ vanishes because $f(\pm\infty)=0$.

Solution to Exercise C.5.

$$\begin{aligned}\tilde{f}(-k) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{-i(-k)x}f(x)\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} (e^{-ikx})^* f(x)\mathrm{d}x \\ &= \left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} e^{-ikx}f(x)\mathrm{d}x\right]^* \quad \text{(because } f(x) \text{ is real)} \\ &= \left[\tilde{f}(-k)\right]^*.\end{aligned}$$

Solution to Exercise C.6. In order to calculate the integral

$$\mathcal{F}[e^{-x^2/b^2}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}e^{-x^2/b^2}\mathrm{d}x \tag{D.2}$$

we express the exponent in Eq. (D.2) as a quadratic function of x and then apply Eq. (C.8):

$$\begin{aligned}
\mathcal{F}[e^{-x^2/b^2}] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{b^2}+ikx+\frac{k^2b^2}{4}-\frac{k^2b^2}{4}\right)}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x}{b}+\frac{kb}{2}\right)^2}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}\int_{-\infty}^{+\infty} e^{-\frac{\left(x+kb^2/2\right)^2}{b^2}}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2b^2}{4}}b\sqrt{\pi} \\
&= \frac{b}{\sqrt{2}}e^{-\frac{k^2b^2}{4}}.
\end{aligned}$$

Solution to Exercise C.7.

a) The result (C.14) can be rewritten as

$$\mathcal{F}[\frac{1}{b\sqrt{\pi}}e^{-x^2/b^2}] = \frac{1}{\sqrt{2\pi}}e^{-k^2b^2/4}. \tag{D.3}$$

In the limit $b \to 0$, the above equation becomes

$$\mathcal{F}[\delta(x)] \overset{(C.9)}{=} \lim_{b\to 0}\mathcal{F}[\frac{1}{b\sqrt{\pi}}e^{-x^2/b^2}] = \lim_{b\to 0}\frac{1}{\sqrt{2\pi}}e^{-k^2b^2/4} = \frac{1}{\sqrt{2\pi}}.$$

b) Equation (C.9) can be rewritten in the form

$$\frac{b}{\sqrt{\pi}}e^{-b^2x^2} \to \delta(x) \qquad \text{for} \qquad b\to\infty. \tag{D.4}$$

We thus find, in the limit $b \to \infty$,

$$\mathcal{F}[1] = \lim_{b\to\infty}\mathcal{F}[e^{-x^2/b^2}] \overset{(C.14)}{=} \lim_{b\to\infty}\frac{b}{\sqrt{2}}e^{-\frac{k^2b^2}{4}} \overset{(D.4)}{=} \frac{\sqrt{\pi}}{\sqrt{2}}\,\delta\left(\frac{k}{2}\right) \overset{(C.4)}{=} \sqrt{2\pi}\,\delta(k).$$

Solution to Exercise C.8. We notice that the required integral is, up to a factor of $\sqrt{2\pi}$, the Fourier transform of the function $f(x) = 1$ at the point $k = -k_0$. Applying Eq. (C.16), we find

$$\int\limits_{-\infty}^{+\infty} e^{ik_0x}dx = \sqrt{2\pi}\mathcal{F}[1](-k_0) = 2\pi\,\delta(-k_0) = 2\pi\,\delta(k_0).$$

Solution to Exercise C.9.

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}[f]](x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x')e^{-ikx'}\mathrm{d}x'\right)e^{ikx}\mathrm{d}k \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x')e^{ik(x-x')}\mathrm{d}k\,\mathrm{d}x' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\left[\int_{-\infty}^{+\infty} e^{ik(x-x')}\mathrm{d}k\right]\mathrm{d}x' \\
&\overset{(C.17)}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x')\left[2\pi\delta(x-x')\right]\mathrm{d}x' \\
&\overset{(C.3)}{=} f(x).
\end{aligned}$$

Solution to Exercise C.10. Recall the definition (C.19) of the inverse Fourier transformation:

$$\mathcal{F}^{-1}[f](k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{ikx} f(x)\mathrm{d}x.$$

[here we exchanged the variables x and k with respect to Eq. (C.19)]. Now replacing the integration variable $x \to -x$ we find

$$\mathcal{F}^{-1}[f](k) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{-ikx} f(-x)\mathrm{d}x = \mathcal{F}[f](-k).$$

Solution to Exercise C.11.

a) We introduce a new integration variable $t = ax$ and act by analogy to Ex. C.1(c)

$$\begin{aligned}
\mathcal{F}[f(ax)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(ax)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(ax)e^{-i\frac{k}{a}ax}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\,|a|} \int_{-\infty}^{+\infty} f(t)e^{-i\frac{k}{a}t}\mathrm{d}t \\
&= \frac{1}{|a|}\tilde{f}(k/a).
\end{aligned}$$

b) Similarly to the above, replacing the integration variable according to $t = x - a$, we have

$$\begin{aligned}
\mathcal{F}[f(x-a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-a)e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-a)e^{-ik(x-a)-ika}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}e^{-ika} \int_{-\infty}^{+\infty} f(t)e^{-ikt}\mathrm{d}t \\
&= e^{-ika}\tilde{f}(k).
\end{aligned}$$

c) Reversing the argument of part (b), we find

$$\begin{aligned}
\mathcal{F}[e^{i\xi x}f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-ikx}e^{i\xi x}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-i(k-\xi)x}\mathrm{d}x \\
&= \tilde{f}(k-\xi).
\end{aligned}$$

d) We use integration by parts and assune that $f(x)$ is a smooth function approaching zero at $\pm\infty$:

$$\begin{aligned}
\mathcal{F}[\mathrm{d}f(x)/\mathrm{d}x] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}f(x)}{\mathrm{d}x} e^{-ikx}\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} f(x)e^{-ikx}\Big|_{-\infty}^{+\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\frac{\mathrm{d}e^{-ikx}}{\mathrm{d}x}\mathrm{d}x \\
&= 0 - (-ik)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x \\
&= ik\tilde{f}(k).
\end{aligned}$$

For the inverse Fourier transformation, we replace the factor of e^{-ikx} by e^{ikx} in the above integrals and obtain

$$\mathcal{F}^{-1}[g(ax)] = \breve{g}(k/a)/|a|; \tag{D.5}$$
$$\mathcal{F}^{-1}[g(x-a)] = e^{ika}\breve{g}(k); \tag{D.6}$$
$$\mathcal{F}^{-1}[e^{i\xi x}g(x)] = \breve{g}(k+\xi); \tag{D.7}$$
$$\mathcal{F}^{-1}[dg(x)/dx] = -ik\breve{g}(k). \tag{D.8}$$

Solution to Exercise C.12. Left for the reader as an independent exercise.

Solution to Exercise C.13. We replace the integration variable according to $y = x - z$. Then we have $\mathrm{d}y = -\mathrm{d}z$ and

$$\begin{aligned}
[f * g](x) &= \int_{-\infty}^{+\infty} f(x-y)g(y)\mathrm{d}y \\
&= -\int_{+\infty}^{-\infty} f(z)g(x-z)\mathrm{d}z \\
&= \int_{-\infty}^{+\infty} f(z)g(x-z)\mathrm{d}z \\
&= [g * f](x).
\end{aligned}$$

Solution to Exercise C.14. According to Eq. (C.3),

$$[\delta * f](x) = \int_{-\infty}^{+\infty} \delta(x-y)f(y)\mathrm{d}y = f(x)$$

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Solution to Exercise C.15.

a)

$$\begin{aligned}
\mathcal{F}[f] \times \mathcal{F}[g] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x \int_{-\infty}^{+\infty} g(x')e^{-ikx'}\mathrm{d}x' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x)g(x')e^{-ik(x+x')}\mathrm{d}x'\right]\mathrm{d}x
\end{aligned}$$

To calculate the integral in brackets, we replace the integration variable according to $t = x + x'$ and obtain

$$\begin{aligned}
\mathcal{F}[f] \times \mathcal{F}[g] &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x)g(t-x)e^{-ikt}\mathrm{d}t\right]\mathrm{d}x \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(x)g(t-x)\mathrm{d}x\right]e^{-ikt}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} [f * g](t)e^{-ikt}\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}}\mathcal{F}[f * g].
\end{aligned}$$

b) Because the inverse Fourier transform is very similar to the direct Fourier transform [see Eq. (C.21)], we can extend the result of part (a) as follows

$$\mathcal{F}^{-1}[f * g] = \sqrt{2\pi}\,\mathcal{F}^{-1}[f] \times \mathcal{F}^{-1}[g]; \tag{D.9}$$

We now apply the inverse Fourier transform to the right-hand side of Eq. (C.28) and obtain, according to the above:

$$\mathcal{F}^{-1}\left[\mathcal{F}[f] * \mathcal{F}[g]\right] = \sqrt{2\pi}\,\mathcal{F}^{-1}\left[\mathcal{F}[f]\right] \times \mathcal{F}^{-1}\left[\mathcal{F}[g]\right] = \sqrt{2\pi}\, f \times g.$$

Now acting with the direct Fourier transform to both sides of the above equation, we arrive at the desired Eq. (C.28).

Solution to Exercise C.16. Left for the reader as an independent exercise.