# Chapter 9

# Solutions to chapter 4 problems

**Solution to Exercise 4.7.** For example, the $x$ component of the angular momentum is defined as $\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$. The position and momentum observables are Hermitian; in addition, we have $[\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_y] = 0$. We can thus write for the Hermitian conjugate of $\hat{L}_x$

$$\hat{L}_x^\dagger \overset{\text{Ex. 1.63}}{=} \hat{p}_z^\dagger \hat{y}^\dagger - \hat{p}_y^\dagger \hat{z}^\dagger = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \hat{L}_x.$$

**Solution to Exercise 4.10.**

a) We use $\hat{L}_j = \epsilon_{jmn}\hat{r}_m\hat{p}_n$ and $[\hat{r}_j, \hat{p}_k] = i\hbar\delta_{jk}$ to write

$$\begin{aligned}
[\hat{L}_j, \hat{r}_k] &= [\epsilon_{jmn}\hat{r}_m\hat{p}_n, \hat{r}_k] \\
&= \epsilon_{jmn}\hat{r}_m[\hat{p}_n, \hat{r}_k] \qquad (\epsilon_{jmn} \text{ and } \hat{r}_m \text{ can be factored out because they commute with } \hat{r}_k) \\
&= \epsilon_{jmn}\hat{r}_m(-i\hbar)\delta_{nk} \\
&= (-i\hbar)\epsilon_{jmk}\hat{r}_m \\
&= i\hbar\epsilon_{jkm}\hat{r}_m. \qquad (\epsilon_{jkm} \text{ is an antisymmetric tensor, so } \epsilon_{jkm} = -\epsilon_{jmk})
\end{aligned}$$

b) Similarly,

$$[\hat{L}_j, \hat{p}_k] = [\epsilon_{jmn}\hat{r}_m\hat{p}_n, \hat{p}_k] = \epsilon_{jmn}\hat{p}_n[\hat{r}_m, \hat{p}_k] = \epsilon_{jmn}\hat{p}_n(i\hbar)\delta_{mk} = i\hbar\epsilon_{jkn}\hat{p}_n.$$

c)

$$\begin{aligned}
[\hat{L}_j, \hat{L}_k] &= [\epsilon_{jmn}\hat{r}_m\hat{p}_n, \epsilon_{klq}\hat{r}_l\hat{p}_q] \\
&= \epsilon_{jmn}\epsilon_{klq}[\hat{r}_m\hat{p}_n, \hat{r}_l\hat{p}_q] \\
&\overset{(1.46)}{=} \epsilon_{jmn}\epsilon_{klq}\big(\hat{r}_m[\hat{p}_n, \hat{r}_l]\hat{p}_q + \hat{r}_l[\hat{r}_m, \hat{p}_q]\hat{p}_n\big) \\
&= \epsilon_{jmn}\epsilon_{klq}\big((-i\hbar)\delta_{nl}\hat{r}_m\hat{p}_q + (i\hbar)\delta_{mq}\hat{r}_l\hat{p}_n\big) \\
&= -i\hbar\epsilon_{jml}\epsilon_{klq}\hat{r}_m\hat{p}_q + i\hbar\epsilon_{jmn}\epsilon_{klm}\hat{r}_l\hat{p}_n \\
&= -i\hbar\epsilon_{ljm}\epsilon_{lqk}\hat{r}_m\hat{p}_q + i\hbar\epsilon_{mnj}\epsilon_{mkl}\hat{r}_l\hat{p}_n \\
&\overset{\text{Ex. 4.8}}{=} -i\hbar(\delta_{jq}\delta_{mk} - \delta_{jk}\delta_{mq})\hat{r}_m\hat{p}_q + i\hbar(\delta_{kn}\delta_{jl} - \delta_{nl}\delta_{jk})\hat{r}_l\hat{p}_n \\
&= -i\hbar\hat{r}_k\hat{p}_j + i\hbar\delta_{jk}\hat{r}_m\hat{p}_m + i\hbar\hat{r}_j\hat{p}_k - i\hbar\delta_{jk}\hat{r}_l\hat{p}_l \\
&= -i\hbar\hat{r}_k\hat{p}_j + i\hbar\hat{r}_j\hat{p}_k.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
i\hbar\epsilon_{jkl}\hat{L}_l &= i\hbar\epsilon_{jkl}\epsilon_{lmn}\hat{r}_m\hat{p}_n \\
&= i\hbar\epsilon_{ljk}\epsilon_{lmn}\hat{r}_m\hat{p}_n \\
&\overset{\text{Ex. 4.8}}{=} i\hbar(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\hat{r}_m\hat{p}_n \\
&= i\hbar\hat{r}_j\hat{p}_k - i\hbar\hat{r}_k\hat{p}_j.
\end{aligned}$$

Comparing the two expressions above, we obtain the desired result: $[\hat{L}_j, \hat{L}_k] = i\hbar\epsilon_{jkl}\hat{L}_l$;

d) Here we use that the square of a vector is its inner product with itself: $\hat{r}^2 = \hat{r}_m\hat{r}_m$. Therefore,

$$[\hat{L}_j, \hat{r}_k\hat{r}_k] \stackrel{(1.46)}{=} \hat{r}_k[\hat{L}_j, \hat{r}_k] + [\hat{L}_j, \hat{r}_k]\hat{r}_k \stackrel{\text{Ex. 4.10(a)}}{=} 2\hat{r}_k\epsilon_{jkl}\hat{r}_l$$

This expression is symmetric with respect to indices $k$ and $l$. On the other hand $\epsilon_{jkl}$ is an antisymmetric tensor (i.e. it changes sign when two indices are interchanged). We thus conclude that $[\hat{L}_j, \hat{r}^2] = 0$.

e) The argument is analogous to part (d):

$$[\hat{L}_j, \hat{p}_k\hat{p}_k] = \hat{p}_k[\hat{L}_j, \hat{p}_k] + [\hat{L}_j, \hat{p}_k]\hat{p}_k = 2\hat{p}_k\epsilon_{jkl}\hat{p}_l = 0$$

f) Again,

$$[\hat{L}_j, \hat{L}_k\hat{L}_k] = \hat{L}_k[\hat{L}_j, \hat{L}_k] + [\hat{L}_j, \hat{L}_k]\hat{L}_k = 2\hat{L}_k\epsilon_{jkl}\hat{L}_l = 0$$

**Solution to Exercise 4.13.** The expectation value of the operator $\hat{L}^2$ in its eigenstate $|\lambda m\rangle$ equals $\langle\lambda m|\,\hat{L}^2\,|\lambda m\rangle = \hbar^2\lambda$. On the other hand, because $|\lambda m\rangle$ is also an eigenstate of $\hat{L}_z$ with eigenvalue $\hbar m$, this expectation value equals

$$\langle\lambda m|\,\hat{L}^2\,|\lambda m\rangle = \langle\lambda m|\left(\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2\right)|\lambda m\rangle = \hbar^2 m^2 + \langle\lambda m|\left(\hat{L}_x^2 + \hat{L}_y^2\right)|\lambda m\rangle\,.$$

The expectation value of the of the operator $\hat{L}_x^2 + \hat{L}_y^2$ is then $\hbar^2(\lambda - m^2)$. Because a square of a Hermitian operator cannot have a negative expectation value[1], we must have $m^2 \leq \lambda$.

**Solution to Exercise 4.14.**

a) The angular momentum components are Hermitian operators, so $\hat{L}_x^\dagger = \hat{L}_x$ and $(i\hat{L}_y)^\dagger = -i\hat{L}_y$. Therefore, $\hat{L}_+^\dagger = (\hat{L}_x + i\hat{L}_y)^\dagger = \hat{L}_x - i\hat{L}_y = \hat{L}_-$.

b) Using the result of Ex. 4.10, we find

$$[\hat{L}_z, \hat{L}_\pm] = [\hat{L}_z, \hat{L}_x \pm i\hat{L}_y] = i\hbar L_y \pm i(-i\hbar)\hat{L}_x = \hbar(\pm\hat{L}_x + i\hat{L}_y) = \pm\hbar\hat{L}_\pm;$$

$$[\hat{L}^2, \hat{L}_\pm] = [\hat{L}^2, \hat{L}_x \pm i\hat{L}_y] = [\hat{L}^2, \hat{L}_x] \pm i[\hat{L}^2, \hat{L}_y] = 0;$$

$$[\hat{L}_+, \hat{L}_-] = [\hat{L}_x + i\hat{L}_y, \hat{L}_x - i\hat{L}_y] = i[\hat{L}_y, \hat{L}_x] - i[\hat{L}_x, \hat{L}_y] = 2\hat{L}_z;$$

c) From

$$\hat{L}_+\hat{L}_- = (\hat{L}_x + i\hat{L}_y)(\hat{L}_x - i\hat{L}_y) = \hat{L}_x^2 + \hat{L}_y^2 - i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) = \hat{L}^2 - \hat{L}_z^2 - i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z;$$

$$\hat{L}_-\hat{L}_+ = (\hat{L}_x - i\hat{L}_y)(\hat{L}_x + i\hat{L}_y) = \hat{L}_x^2 + \hat{L}_y^2 + i(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x) = \hat{L}^2 - \hat{L}_z^2 + i[\hat{L}_x, \hat{L}_y] = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$$

we find the required relation

$$\hat{L}^2 = \hat{L}_+\hat{L}_- + \hat{L}_z^2 - \hbar\hat{L}_z = \hat{L}_-\hat{L}_+ + \hat{L}_z^2 + \hbar\hat{L}_z.$$

**Solution to Exercise 4.15.**

[1] This general statement can be proven as follows. Consider some Hermitian operator $\hat{A}$. The expectation value of $\hat{A}^2$ in an arbitrary quantum state $|\psi\rangle$ equals

$$\left\langle\psi\right|\hat{A}^2\left|\psi\right\rangle = \left\langle\psi\right|\hat{A}\hat{\mathbf{1}}\hat{A}\left|\psi\right\rangle = \sum_i \left\langle\psi\right|\hat{A}\left|v_i\right\rangle\left\langle v_i\right|\hat{A}\left|\psi\right\rangle = \sum_i \left|\left\langle\psi\right|\hat{A}\left|v_i\right\rangle\right|^2 \geq 0,$$

where $\{|v_i\rangle\}$ is an arbitrary orthonormal basis.

a) In order to verify if the state $\hat{L}_+ |\lambda m\rangle$ is an eigenstate of $\hat{L}^2$ and $\hat{L}_z$, let us subject this state to the action of these operators. Because $\hat{L}^2$ commutes with $\hat{L}_+$, we have

$$\hat{L}^2 \hat{L}_+ |\lambda m\rangle = \hat{L}_+ \hat{L}^2 |\lambda m\rangle = \hat{L}_+ \hbar^2 \lambda |\lambda m\rangle = \hbar^2 \lambda \hat{L}_+ |\lambda m\rangle .$$

In other words, $\hat{L}_+ |\lambda m\rangle$ is an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2\lambda$.

To perform a similar calculation for $\hat{L}_z$, we rewrite the expression for the commutator of $\hat{L}_z$ and $\hat{L}_+$ obtained in Ex. 4.14 as follows:

$$\hat{L}_z \hat{L}_+ = \hat{L}_+ \hat{L}_z + \hbar \hat{L}_+,$$

and thus

$$\hat{L}_z \hat{L}_+ |\lambda m\rangle = \left(\hat{L}_+ \hat{L}_z + \hbar \hat{L}_+\right) |\lambda m\rangle = \left(\hbar m \hat{L}_+ + \hbar \hat{L}_+\right) |\lambda m\rangle = \hbar(m+1) \hat{L}_+ |\lambda m\rangle .$$

We see that the action of the operator $\hat{L}_z$ on the state $\hat{L}_+ |\lambda m\rangle$ is equivalent to multiplying this state by $\hbar(m+1)$, so $\hat{L}_+ |\lambda m\rangle$ is an eigenstate of $\hat{L}_z$ with eigenvalue $\hbar(m+1)$.

b) Similarly, because

$$\hat{L}_z \hat{L}_- = \hat{L}_- \hat{L}_z - \hbar \hat{L}_-,$$

we have

$$\hat{L}_z \hat{L}_- |\lambda m\rangle = \left(\hat{L}_- \hat{L}_z - \hbar \hat{L}_-\right) |\lambda m\rangle = \left(\hbar m \hat{L}_- - \hbar \hat{L}_-\right) |\lambda m\rangle = \hbar(m-1) \hat{L}_- |\lambda m\rangle ,$$

so $\hat{L}_- |\lambda m\rangle$ is an eigenstate of $\hat{L}_z$ with eigenvalue $\hbar(m-1)$. The proof for $\hat{L}^2$ is analogous to part (a).

**Solution to Exercise 4.16.**

a) Let $|\psi\rangle = \hat{L}_+ |\lambda m\rangle$. From the previous exercise, we know that $|\psi\rangle$ is an eigenstate of $\hat{L}_z$ with eigenvalue $\hbar(m+1)$, i.e. $|\psi\rangle = A |\lambda, m+1\rangle$, where $A$ is some constant. We need to find $A$. To this end, we notice that $\langle\psi| = \langle\lambda m| \hat{L}_+^\dagger = \langle\lambda m| \hat{L}_-$ and calculate

$$\langle\psi|\,\psi\rangle = \left\langle\lambda m\right|\,\hat{L}_- \hat{L}_+ \left|\,\lambda m\right\rangle \overset{\text{Ex. 4.14(c)}}{=} \left\langle\lambda m\right|\,\hat{L}^2 - \hat{L}_z^2 - \hbar \hat{L}_z \left|\,\lambda m\right\rangle = \hbar^2(\lambda - m^2 - m)$$

(in the last equality, we used the fact that $|\lambda m\rangle$ is an eigenstate of both $\hat{L}^2$ and $\hat{L}_z$). But on the other hand,

$$\langle\psi|\,\psi\rangle = |A|^2 \langle\lambda, m+1|\,\lambda, m+1\rangle = |A|^2,$$

because the eigenstates of the angular momentum operator are normalized. Arbitrarily choosing the phase of $A$ equal to zero, we find $A = \hbar\sqrt{\lambda - m(m+1)}$.

b) Similarly, if $|\phi\rangle = \hat{L}_- |\lambda m\rangle$, then, on one hand,

$$\langle\phi|\,\phi\rangle = \left\langle\lambda m\right|\,\hat{L}_+ \hat{L}_- \left|\,\lambda m\right\rangle = \left\langle\lambda m\right|\,\hat{L}^2 - \hat{L}_z^2 + \hbar \hat{L}_z \left|\,\lambda m\right\rangle = \hbar^2(\lambda - m^2 + m)$$

and on the other hand

$$\langle\phi|\,\phi\rangle = |B|^2 \langle\lambda, m-1|\,\lambda, m-1\rangle = |B|^2.$$

Therefore, $B = \hbar\sqrt{\lambda - m(m-1)}$.

**Solution to Exercise 4.18.** First, because the state $|\lambda m\rangle$ is an eigenstate of both $\hat{L}^2$ and $\hat{L}_z$, we have

$$\left\langle lm\right|\,\hat{L}^2\left|\,l'm'\right\rangle = \hbar^2 l'(l'+1) \langle lm|\,l'm'\rangle ; \tag{9.1}$$

$$\left\langle lm \right| \hat{L}_z \left| l'm' \right\rangle = \hbar m' \left\langle lm \right| l'm' \rangle ; \tag{9.2}$$

Both these expressions vanish if $l \neq l'$.

Second, the action of the raising and lowering operators on states $|\lambda m\rangle$ is known from Ex. 4.16:

$$\left\langle lm \right| \hat{L}_\pm \left| l'm' \right\rangle = \hbar\sqrt{l'(l'+1) - m'(m' \pm 1)} \left\langle lm \right| l', m' \pm 1 \rangle , \tag{9.3}$$

so these operators also preserve $l$.

Finally, the $x$- and $y$-components of the angular momentum can be written as linear combinations of the raising and lowering operators according to the definition 4.2 of the latter:

$$\hat{L}_x = \frac{L_+ + L_-}{2}; \tag{9.4}$$

$$\hat{L}_y = \frac{L_+ - L_-}{2i} \tag{9.5}$$

and hence they must preserve the value of $l$ as well.

**Solution to Exercise 4.19.** In the case $l = 1/2$, the eigenvalue of $m$ can be either $+1/2$ or $-1/2$. Hence the matrices are two-dimensional. The matrix elements can be found according to Eqs. (9.1)–(9.5). For $\hat{L}^2$ we have

$$\left\langle \frac{1}{2}, m \right| \hat{L}^2 \left| \frac{1}{2}, m' \right\rangle = \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right) \left\langle \frac{1}{2}, m \right| \left. \frac{1}{2}, m' \right\rangle = \frac{3}{4}\hbar^2 \delta_{mm'},$$

so the matrix is

$$\hat{L}^2 \leftrightarrow \frac{3\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

According to (9.2), the matrix element of $\hat{L}_z$ is

$$\left\langle \frac{1}{2}, m \right| \hat{L}_z \left| \frac{1}{2}, m' \right\rangle = \hbar m' \left\langle \frac{1}{2}, m \right| \left. \frac{1}{2}, m' \right\rangle = \hbar m' \delta_{mm'},$$

so the matrix is

$$\hat{L}_z \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Our next step is to find the matrices of the raising and lowering operators. According to Eq. (9.3), nonvanishing matrix elements of $\hat{L}_+$ have $m = m' + 1$. If $l = l' = 1/2$, the only possibility to satisfy this condition is to set $m = 1/2$, $m' = -1/2$. Then $\sqrt{l'(l'+1) - m'(m' \pm 1)} = 1$ and

$$\hat{L}_+ \leftrightarrow \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By the same token,

$$\hat{L}_- \leftrightarrow \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now, using Eqs, (9.4) and (9.5), we find

$$\hat{L}_x \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{L}_y \leftrightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

**Solution to Exercise 4.38.** We first notice that the operator $\hat{S}_{\vec{R}}$ for a spin-1/2 particle can be written as

$$\hat{S}_{\vec{R}} = \vec{\hat{S}} \cdot \vec{R} \overset{(4.14)}{=} \frac{\hbar}{2}\vec{\hat{\sigma}} \cdot \vec{R},$$

where $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the "vector" consisting of Pauli operators and $\vec{R}$ is a unit length vector. From Ex. 1.88 we know that the operator $\hat{\vec{\sigma}} \cdot \vec{R}$ has eigenvalues $\lambda_{1,2} = \pm 1$ and hence the eigenvalues of $\hat{S}_{\vec{R}}$ are $\pm\hbar/2$.

Our goal is to find the eigenstate of this operator with the eigenvalue $+\hbar/2$. We begin by expressing $\hat{S}_{\vec{R}}$ in the matrix form in the canonical basis[2]:

$$\begin{aligned}
\hat{S}_{\vec{R}} &= \sin\theta_0 \cos\phi_0 \hat{S}_x + \sin\theta_0 \sin\phi_0 \hat{S}_y + \cos\theta_0 \hat{S}_z \\
&\overset{(4.14)}{=} \frac{\hbar}{2}\left[\sin\theta_0\cos\phi_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta_0 \sin\phi_0 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\theta_0 & \sin\theta_0\cos\phi_0 - i\sin\theta_0\sin\phi_0 \\ \sin\theta_0\cos\phi_0 + i\sin\theta_0\sin\phi_0 & -\cos\theta_0 \end{pmatrix}.
\end{aligned}$$

After some straightforward transformations, we find the normalized eigenstate:

$$\left|m_{s_{\vec{R}}} = 1/2\right\rangle = \begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i\phi_0} \end{pmatrix}. \tag{9.6}$$

Let us calculate this eigenstate for a few special cases:

| direction of $\vec{R}$ | $\theta_0$, $\phi_0$ | eigenstate |
|---|---|---|
| $+z$ | $0,\ 0$ | $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $-z$ | $\pi,\ 0$ | $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| $+x$ | $\pi/2,\ 0$ | $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ |
| $-x$ | $\pi/2,\ \pi$ | $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ |
| $+y$ | $\pi/2,\ \pi/2$ | $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ |
| $-y$ | $\pi/2,\ -\pi/2$ | $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ |

Of course, we have already determined these eigenstates of Pauli operators in Chapter 1.

**Solution to Exercise 4.41.** Let $\omega$ be the angular frequency of the particle's orbital motion. The particle makes a full circle in time $T = 2\pi/\omega$, and the current associated with this motion is thus $I = e/T = e\omega/2\pi$. The area of the orbit is $A = \pi r^2$, where $r$ is the radius. Substituting these quantities into Eq. (4.36), we find for the magnetic moment

$$\mu = \frac{e\omega r^2}{2c}.$$

On the other hand, the mechanical angular momentum of the orbiting particle is $L = m\omega r^2$. The magnetic moment can thus be expressed as

$$\mu = L\frac{e}{2mc}.$$

Both the angular momentum and the magnetic moment are actually vectors directed orthogonally out of the plane of the orbit. Therefore, the above expression is also valid in its vector form.

**Solution to Exercise 4.42.** Equation (4.37) holds for all three components of the angular momentum; in particular, the $z$-component:

$$\mu_z = L_z\frac{e}{2mc}.$$

[2]Under the *canoncial* basis of the Hilbert space of a particle's spin states we understand the basis of eigenstates $|sm_s\rangle$ of operators $\hat{S}^2$ and $\hat{S}_z$.

The state with a definite magnetic quantum number $m_l$ is $|lm_l\rangle$, an eigenstate of $\hat{L}_z$ with eigenvalue $L_z = \hbar m_l$. We can thus write the $z$ component of the magnetic moment as

$$\mu_z = \frac{e}{2mc}\hbar m_l.$$

**Solution to Exercise 4.43.** According to Ex. 4.40, the interaction Hamiltonian equals $\hat{H} = -\vec{\hat{\mu}}\cdot\vec{B}$. Because the magnetic field is in the $y$ direction, we can write

$$H = -\hat{\mu}_y B \overset{(4.40)}{=} -\frac{ge}{2mc}\hat{S}_y.$$

It is convenient to solve this problem in the matrix notation in the canonical basis. Because [Ex. 4.10(f)] the Hamiltonian commutes with $\hat{S}^2$, the absolute value of the spin $s = 1/2$ is preserved, so we can restrict our Hilbert space to the subspace spanned by states $|s = 1/2, m_s = 1/2\rangle$ and $|s = 1/2, m_s = -1/2\rangle$. In this subspace, the initial state has the matrix

$$|\psi(0)\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the Hamiltonian, according to Eq. (4.14), takes the form

$$\hat{H} = -\hat{\mu}_y B \leftrightarrow -\frac{geB}{2mc}\frac{\hbar}{2}\hat{\sigma}_y.$$

The evolution of the electron's spin is than governed by the Schrödinger equation

$$\left|\dot{\psi}(t)\right\rangle = -\frac{i}{\hbar}\hat{H}\,|\psi\rangle\,,$$

whose solution is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\psi(0)\rangle = e^{i\frac{geBt}{4mc}\hat{\sigma}_y}\,|\psi(0)\rangle\,.$$

This matrix exponent has already been calculated in Ex. 1.95:

$$e^{i\varphi\hat{\sigma}_y} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}.$$

Using this solution, with $\varphi = \frac{geBt}{4mc}$, we find the evolution of the spin:

$$|\psi(t)\rangle \leftrightarrow \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ -\sin\varphi \end{pmatrix}.$$

The Stern-Gerlach measurement constitutes a measurement of $\hat{S}_z$ in the state $|\psi(t)\rangle$. Because this state is already expressed in the canonical basis, we immediately find the probability to detect $m_s = 1/2$ as $\mathrm{pr}_{1/2} = |\langle m_s = 1/2|\,\psi(t)\rangle|^2 = \cos^2\varphi$.

**Solution to Exercise 4.44.** We proceed along the lines of the previous problem's solution, but the Hamiltonian is now

$$H = -\vec{\hat{\mu}}\cdot\vec{B} = -\frac{ge}{2mc}\vec{\hat{S}}\cdot\vec{B} = -\frac{ge\hbar}{4mc}\vec{\hat{\sigma}}\cdot\vec{B},$$

where $\vec{\hat{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the "vector" consisting of Pauli operators. The evolution under this Hamiltonian is then given by

$$e^{-\frac{i}{\hbar}\hat{H}t} = e^{i\frac{get}{4mc}\vec{\hat{\sigma}}\cdot\vec{B}} = e^{i\varphi\vec{\hat{\sigma}}\cdot\vec{v}},$$

where we have defined $\varphi = \frac{geBt}{4mc}$ and $\vec{v} = (\sin\theta_0, 0, \cos\theta_0)$ is the unit length vector in the direction of the magnetic field.

Now we are in position to apply the result of Ex. 1.88. We find

$$\begin{aligned} e^{-\frac{i}{\hbar}\hat{H}t} &= \cos\varphi\hat{\mathbf{1}} + i\sin\varphi\vec{v}\cdot\vec{\hat{\sigma}} \\ &= \cos\varphi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sin\varphi\left[\sin\theta_0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cos\theta_0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &= \begin{pmatrix} \cos\varphi + i\sin\varphi\cos\theta_0 & i\sin\varphi\sin\theta_0 \\ i\sin\varphi\sin\theta_0 & \cos\varphi - i\sin\varphi\cos\theta_0 \end{pmatrix}. \end{aligned}$$

Applying this evolution operator to the initial state $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = \begin{pmatrix} \cos\varphi + i\sin\varphi\cos\theta_0 \\ i\sin\varphi\sin\theta_0 \end{pmatrix}.$$

When this state is subjected to the Stern-Gerlach measurement, the probability to detect $m_s = 1/2$ is

$$\mathrm{pr}_{1/2} = |\langle m_s = 1/2|\,\psi(t)\rangle|^2 = \cos^2\varphi + \sin^2\varphi\cos^2\theta_0.$$

**Solution to Exercise 4.45.** The Hamiltonian associated with the magnetic field along the $z$ axis is given by

$$H = -\hat{\mu}_z B = -\frac{\hbar geB}{4mc}\hat{\sigma}_z$$

and the associated evolution (in the canonical basis)

$$e^{-\frac{i}{\hbar}\hat{H}t} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix},$$

with $\varphi = \frac{geBt}{4mc}$.

Applying this evolution to the eigenstate (9.6) of the spin $\hat{S}_{\vec{R}}$ oriented along vector $\vec{R}$ characterized by polar angles $(\theta_0, \phi_0)$, we obtain

$$\begin{aligned} |\psi(t)\rangle &= e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle \\ &= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}\begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i\phi_0} \end{pmatrix} \\ &= e^{i\varphi}\begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2)e^{i(\phi_0-2\varphi)} \end{pmatrix}. \end{aligned}$$

Comparing this result with Eq. (9.6), we find that the state after the evolution is physically equivalent to an eigenstate of the spin $\hat{S}_{\vec{R}'}$ with $\vec{R}'$ characterized by polar angles $(\theta_0, \phi_0 - 2\varphi)$. In other words, the spin has precessed by the angle $2\varphi = \frac{geBt}{2mc}$.

Note that we can write $2\varphi = \Omega_L t$, where $\Omega_L = \frac{geB}{2mc}$ is the Larmor frequency, which is known to determine the precession of magnetic moments in classical physics.

**Solution to Exercise 4.46.** The Stern-Gerlach measurement is that of the spin component $\hat{S}_{\vec{R}}$ with $\vec{R}$ characterized by polar angles $(\theta_0, 0)$. The probabilities of possible measurement outcomes are given by the Second postulate of quantum mechanics: $\mathrm{pr}_i = |\langle\psi|\,v_i\rangle|^2$, where $|\psi\rangle$ is the input state, whose canonical representation is $|\psi\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|v_i\rangle$ are the eigenstates of $\hat{S}_{\vec{R}}$. In Ex. 4.38 we have found one of these states:

$$\left|m_{s_{\vec{R}}} = 1/2\right\rangle = \begin{pmatrix} \cos(\theta_0/2) \\ \sin(\theta_0/2) \end{pmatrix}.$$

The associated detection probability is then

$$\mathrm{pr}_{1/2} = \sin^2(\theta_0/2).$$

The probability of the other measurement result is

$$\mathrm{pr}_{-1/2} = 1 - \mathrm{pr}_{1/2} = \cos^2(\theta_0/2).$$