Chapter 6

Solutions to chapter 1 problems

Solution to Exercise 1.1.

a) *Yes. No. Yes. Yes.* A field over itself is a linear space because all properties listed in Definition 1.1 follow from the properties of addition and multiplication of the field elements. $\mathbb{R}$ over $\mathbb{C}$ is not a linear space because when you multiply a "vector" (real number) by a "scalar" (complex number) we may obtain a number that is not real, i.e. no longer an element of the linear space. Finally, $\mathbb{C}$ over $\mathbb{R}$ is a linear space because addition of complex numbers and multiplication of a complex number by a real number yields a complex number, which proves that these operations are defined correctly. Their properties can be readily verified to be equivalent to the axioms in Definition 1.1.

b) *Yes. Yes. No.* Addition of two polynomials or their multiplication by a number (either real or complex) will yield a polynomial of power that is not higher than the original ones. Polynomials of power $> n$ do not form a linear space, e.g. because it does not contain a zero element.

c) *Yes. No.* In the first case, the zero element is $f(x) \equiv 0$. The set of functions such that $f(1) = 1$ does not contain this element.

d) *Yes.* It is known from geometry that addition of vectors and multiplication of a vector by a number yields a vector. The properties of these operations can be verified to satisfy the linear space axioms. Note that because an n-dimensional vector can be defined by a column with n real numbers (the coordinates of the vector), we can say that the linear space of n-dimensional geometric vectors is *isomorphic* (equivalent) to the linear space of columns of n real numbers.

Solution to Exercise 1.2.

a) Suppose there are two zero elements, $|\mathrm{zero}\rangle$ and $|\mathrm{zero}'\rangle$. Then, according to Axiom 3, we see that, on one hand, $|\mathrm{zero}\rangle + |\mathrm{zero}'\rangle = |\mathrm{zero}'\rangle$ and on the other hand, $|\mathrm{zero}\rangle + |\mathrm{zero}'\rangle \overset{\mathrm{Ax.1}}{=} |\mathrm{zero}'\rangle + |\mathrm{zero}\rangle = |\mathrm{zero}\rangle$. Therefore, $|\mathrm{zero}\rangle$ and $|\mathrm{zero}'\rangle$ are equal to the same element of $\mathbb{V}$, thus they must be equal to each other.

b) Let us start with $|a\rangle + |x\rangle = |a\rangle$ and add $(-|a\rangle)$ to both sides of the equation. We can transform the left-hand side as follows: $|a\rangle + |x\rangle + (-|a\rangle) \overset{\mathrm{Ax.1,2}}{=} [|a\rangle + (-|a\rangle)] + |x\rangle \overset{\mathrm{Ax.4}}{=} |\mathrm{zero}\rangle + |x\rangle = |x\rangle$. The right-hand side is $|a\rangle + (-|a\rangle) = |\mathrm{zero}\rangle$. The two sides are equal, i.e. $|x\rangle = |\mathrm{zero}\rangle$.

c) $|a\rangle + 0\,|a\rangle \overset{\mathrm{Ax.8}}{=} 1\,|a\rangle + 0\,|a\rangle \overset{\mathrm{Ax.6}}{=} (1+0)\,|a\rangle = 1\,|a\rangle = |a\rangle \quad \overset{\mathrm{Ex.1.2(b)}}{\Longrightarrow} \quad 0\,|a\rangle = |\mathrm{zero}\rangle$.

d) $(-1)\,|a\rangle + |a\rangle \overset{\mathrm{Ax.6,8}}{=} (-1+1)\,|a\rangle = 0\,|a\rangle \overset{\mathrm{Ex.1.2(c)}}{=} 0$

e) $(-|\mathrm{zero}\rangle) + |\mathrm{zero}\rangle \overset{\mathrm{Ax.4}}{=} |\mathrm{zero}\rangle \quad \overset{\mathrm{Ex.1.2(b)}}{\Longrightarrow} \quad -|\mathrm{zero}\rangle = |\mathrm{zero}\rangle$

f) This is because $(-|a\rangle)$ can be written as $(-1)|a\rangle$, and multiplication of a vector by a number yields a unique vector.

g) If $|a\rangle = |b\rangle$ then $|a\rangle - |b\rangle = http : //www.novell.com/common/util/langselect.php?http : //www.novell.com/products/opensuse/?sourceid = suselinux10.1\,|a\rangle - |a\rangle = |a\rangle + (-|a\rangle) = 0$. Conversely, if $|a\rangle - |b\rangle = 0$ then, adding $|b\rangle$ to both sides and using associativity, we find $|a\rangle = |b\rangle$.

Solution to Exercise 1.3. Suppose one of the vectors (without loss of generality we assume it is $|v_1\rangle$) can be expressed as a linear combination of others: $|v_1\rangle = \lambda_2|v_2\rangle + \ldots + \lambda_N|v_N\rangle$. Then the nontrivial linear combination $-|v_1\rangle + \lambda_2|v_2\rangle + \ldots + \lambda_N|v_N\rangle$ equals zero, i.e. the set is not linearly independent. Conversely: suppose there exists a nontrivial linear combination $\lambda_1|v_1\rangle + \ldots + \lambda_N|v_N\rangle$ that is equal to zero. One of the λ's (assume it is λ_1) is not equal to zero. Then we can express $|v_1\rangle = (\lambda_2/\lambda_1)|v_2\rangle + \ldots + (\lambda_N/\lambda_1)|v_N\rangle$.

Solution to Exercise 1.4.

a) Two vectors $|v_1\rangle$ and $|v_2\rangle$ being parallel means that there exists a number λ such that $|v_1\rangle == \lambda|v_2\rangle$. But that also means that one of the vectors can be expressed through the other, i.e. they are not linearly independent.

   For the second part of the question, let us consider three vectors with coordinates $|v_1\rangle = (x_1, y_1)$, $|v_2\rangle = (x_2, y_2)$, $|v_3\rangle = (x_3, y_3)$. Their linear dependence means that, for example, $|v_1\rangle = \lambda_2|v_2\rangle + \lambda_3|v_3\rangle$. This translates into a set of equations

$$\begin{aligned} x_1 &= \lambda_2 x_2 + \lambda_3 x_3; \\ y_1 &= \lambda_2 y_2 + \lambda_3 y_3 \end{aligned} \tag{6.1}$$

   which we solve to obtain

$$\begin{aligned} \lambda_2 &= \frac{x_1y_3 - y_1x_3}{x_2y_3 - x_3y_2}; \\ \lambda_3 &= \frac{x_2y_1 - y_2x_1}{x_2y_3 - x_3y_2}. \end{aligned} \tag{6.2}$$

   This solution exists, meaning that the three vectors are not linearly independent, unless $x_2y_3 - x_3y_3 = 0$, or $x_2/y_2 = x_3/y_3$. The latter case means that $|v_2\rangle$ and $|v_3\rangle$ are parallel to each other, i.e. these two vectors are not linearly independent.

b) The vectors being noncomplanar means that none of them is zero (because a zero vector can be ascribed to any plane). Now let us consider any two of the three vectors, $|v_1\rangle$ and $|v_2\rangle$. These two vectors form a plane, and any linear combination of theirs will lie within that plane. But the third vector, $|v_3\rangle$, is known to lie outside of that plane, and hence cannot be a linear combination of the first two.

Solution to Exercise 1.5. As shown in the solution to Exercise 1.4(a), any two non-parallel vectors are sufficient to express any other vector as their linear combination.

Solution to Exercise 1.6. Suppose there exists a basis $V = \{|v_i\rangle\}$ in $\mathbb{V}$ of power N and there exists another basis $W = \{|w_i\rangle\}$ in $\mathbb{V}$ of power $M > N$. Let us consider a set

$$\{|w_1\rangle, |v_1\rangle, \ldots, |v_N\rangle\}. \tag{6.3}$$

This set must be linearly dependent because $|w_1\rangle$ can be expressed through $|v\rangle$'s:

$$|w_1\rangle = \lambda_1|v_1\rangle + \ldots + \lambda_N|v_N\rangle. \tag{6.4}$$

One of the coefficients in this combination (without loss of generality we assume it is λ_1) must be nonzero. If this is the case, we can express $|v_1\rangle$ through

$$\{|w_1\rangle, |v_2\rangle, \ldots, |v_N\rangle\}, \tag{6.5}$$

and therefore the above set is a spanning set in $\mathbb{V}$. We can repeat this procedure of replacing $|v\rangle$'s by $|w\rangle$'s and show that the set

$$\{|w_1\rangle, |w_2\rangle, \ldots, |w_N\rangle\}, \tag{6.6}$$

is also a spanning set. But then all $|w_i\rangle$ with $N < i \leq M$ can be expressed as a linear combination of $|w_1\rangle, |w_2\rangle, \ldots, |w_N\rangle$, which means that the set W is not linearly independent, i.e. is not a basis, which contradicts our initial assumption.

Solution to Exercise 1.7.

a) Let $A = \{|v_i\rangle\}_{i=1}^N$ be some basis in $\mathbb{V}$. We need to prove that any linearly independent set of $N = \dim \mathbb{V}$ elements is a spanning set. Suppose this is not true for a contradiction. Then let $B = \{|w_j\rangle\}_{j=1}^N$ be another linearly independent set of N vectors, this one not spanning $\mathbb{V}$. Then there are vectors in $\mathbb{V}$ which are not linearly combinations of the $|w_j\rangle$ but are of the $|v_i\rangle$. Hence by appending B with elements of A, we can construct a new set

$$C = \{|w_1\rangle, |w_2\rangle, \ldots, |w_N\rangle, |v_l\rangle, \ldots, |v_m\rangle\} \tag{6.7}$$

where $1 < l < m < N$ (without loss of generality). Since C is linearly independent and spans $\mathbb{V}$, it is a basis of $\mathbb{V}$ of dimension $\dim C > N = \dim A$. This contradicts Exercise 1.6, showing that all linearly independent sets of dimension $\dim \mathbb{V}$ must span $\mathbb{V}$.

b) Suppose there exists a set $A = \{|v_i\rangle\}_{i=1}^N$ of N vectors that is a spanning set in $\mathbb{V}$ but not linearly independent: one of the elements in this set, say, $|v_N\rangle$, can be expressed as a linear combination of $|v_1\rangle, \ldots, |v_{N-1}\rangle$. But this means that these $N-1$ vectors also form a spanning set in $\mathbb{V}$. We repeat this procedure of eliminating linearly dependent vectors from A until we obtain a linearly independent subset of A which contains less than N elements and yet is a spanning set in $\mathbb{V}$. But this means that this subset is a basis in $\mathbb{V}$, which contradicts the result of Ex. 1.6

Solution to Exercise 1.8.

a) See Solution to Exercise 1.4(a).

b) According to Exercise 1.4(b), any three non-complanar vectors form a linearly independent set. Because the space is three-dimensional, we know that any linearly-independent set of three vectors must form a basis.

Solution to Exercise 1.9. Let $\{|w_i\rangle\}_{i=1}^N$ be the basis we are trying to decompose our vector $|v\rangle$ into. Assume for a contradiction that there is more than one decomposition, say

$$\lambda_1 |w_1\rangle + \cdots + \lambda_N |w_N\rangle = |v\rangle = \mu_1 |w_1\rangle + \cdots + \mu_N |w_N\rangle \tag{6.8}$$

where $\lambda_l \neq \mu_l$ for all the l. Then, moving everything to the right hand side of the equation, we get

$$0 = (\lambda_1 - \mu_1) |w_1\rangle + \cdots + (\lambda_N - \mu_N) |w_N\rangle \tag{6.9}$$

where not all the coefficients on the right are equal to zero. The only way this could happen was if our original set of vectors was not linearly independent, in other words, not a basis, in contradiction to our assumptions. Hence there is only one decomposition into each basis for any given vector.

Solution to Exercise 1.10. We must first find a decomposition of $|v_k\rangle$ into the basis, in other words, we must find a set of λ_i such that $|v_k\rangle = \sum_i \lambda_i |v_i\rangle$. Notice, however, that $|v_k\rangle$ is a member of the basis and hence a set of λ_i's that work (and the only set that works according to Exercise 1.9) is just the set where all the λ_i's are zero except the kth one. In matrix form, this becomes

$$|v_k\rangle \leftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{th position} \tag{6.10}$$

Solution to Exercise 1.11. The ordered pair (x, y) can also be written as the 2-vector $\begin{pmatrix} x \\ y \end{pmatrix}$ and hence the following holds:

$$(x, y) \equiv \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.11}$$

This tells us that the pair (x, y) does indeed represent a decomposition into a basis, namely the standard basis of $\mathbb{R}^2$, $\left\{ \vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Solution to Exercise 1.12. In order to show that a set of vectors is a basis of a space $\mathbb{V}$, we must show that it is linearly independent and that it spans $\mathbb{V}$.

a) $\{\vec{a}, \vec{c}\}$ is linearly independent because if we let $0 = \lambda_1 \vec{a} + \lambda_2 \vec{c}$, then we get the following:

$$\begin{aligned} \vec{0} &= \lambda_1 \vec{a} + \lambda_2 \vec{c} \\ \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2\lambda_1 \\ 3\lambda_2 \end{pmatrix} \end{aligned}$$

The only way this can be satisfied is when both λ_1 and λ_2 are equal to zero, and hence the vectors are linearly independent. To see that the set spans the whole xy-plane, we set any arbitrary point of the plane equal to an arbitrary linear combination of $\vec{a}$ and $\vec{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\lambda_1 \\ 3\lambda_2 \end{pmatrix}$$

Solution to Exercise 1.13. We need to apply the definition of the geometric scalar product, $\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$, to verify each of the inner product properties.

1. $\vec{a} \cdot (\vec{b} + \vec{c}) = x_a(x_b + x_c) + y_a(y_b + y_c) = x_a x_b + x_a x_c + y_a y_b + y_a y_c = (x_a x_b + y_a y_b) + (x_a x_c + y_a y_c) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
2. $\vec{a} \cdot (\lambda \vec{b}) = (x_a \lambda x_b + y_a \lambda y_b) = \lambda(x_a x_b + y_a y_b) = \lambda \vec{a} \cdot \vec{b}$
3. $\vec{a} \cdot \vec{b} = (x_a x_b + y_a y_b) = \vec{b} \cdot \vec{a}$ (because this is a real space, conjugation can be omitted)
4. $\vec{a} \cdot \vec{a} = x_a^2 + y_a^2$ is a real number greater or equal to zero. The only possibility for this number to be zero is to have $x_a = y_a = 0$, i.e. $\vec{a} = 0$.

Solution to Exercise 1.14. For $|x\rangle = \sum_i \lambda_i |a_i\rangle$, we find, using Properties 1 and 2 of the inner product, $\langle b|\, x\rangle = \sum_i \langle b| (\lambda_i |a_i\rangle) = \sum_i \lambda_i \langle b|\, a_i\rangle$. According to Property 3, and $\langle x|\, b\rangle = \langle x|\, b\rangle^* = \sum_i \lambda_i^* \langle b|\, a_i\rangle^* = \sum_i \lambda_i^* \langle a_i|\, b\rangle$.

Solution to Exercise 1.15. We write, for arbitrary $|b\rangle$, zero $= |b\rangle - |b\rangle$ and thus, by Property 1, $\langle a|\,\text{zero}\rangle = \langle a|\, b\rangle - \langle a|\, b\rangle = 0$. The reverse inner product $\langle \text{zero}|\, a\rangle$ is then also zero by Property 4.

Solution to Exercise 1.16. Let $\{|v_i\rangle\}_{i=1}^M$ be a set of orthogonal vectors. Suppose these vectors are linearly dependent, i.e. one of them (say $|v_1\rangle$) can be written as a linear combination of others:

$$|v_1\rangle = \sum_{i=2}^{N} a_i |v_i\rangle . \tag{6.12}$$

We take the inner product of both sides of Eq. (6.12) with $|v_1\rangle$. Using Property 1 of the inner product, we find

$$\langle v_1|\, v_1\rangle = \sum_{i=2}^{N} a_i \langle v_1|\, v_i\rangle . \tag{6.13}$$

In the above equation, the left-hand side cannot be zero because of Property 4 of the inner product; the right-hand side is zero due to the ortogonality of the set $\{|v_i\rangle\}$. We arrive at a contradiction.

Solution to Exercise 1.17. Let $|\psi'\rangle = e^{i\phi}|\psi\rangle$. Then, using the result of Ex. 1.14, we write

$$\langle \psi'|\, \psi'\rangle \;=\; (e^{i\phi})^* \langle \psi|\, \psi'\rangle = (e^{-i\phi})(e^{i\phi}) \langle \psi|\, \psi\rangle = \langle \psi|\, \psi\rangle . \tag{6.14}$$

Solution to Exercise 1.18. Let $|\psi'\rangle = e^{i\phi}|\psi\rangle$. Then, using the result of Ex. 1.14, and using the fact that the norm, N, is by definition a real number, we write

$$\begin{aligned} \langle \psi|\, \psi\rangle &= N^2(2\langle \text{alive}|\, \psi\rangle - i\langle \text{alive}|\, \psi\rangle) \\ &= N^2(4\langle \text{alive}|\, \text{alive}\rangle + 2i\langle \text{alive}|\, \text{dead}\rangle) - 2i\langle \text{dead}|\, \text{alive}\rangle) + \langle \text{dead}|\, \text{dead}\rangle . \end{aligned} \tag{6.15}$$

Now since $|\text{dead}\rangle$ and $|\text{alive}\rangle$ are physical states, their inner products with themselves equals 1. On they other hand, these states are incompatible with each other, so the inner product between the two vanishes. Hence we have $\langle \psi|\, \psi\rangle = N^2(4+1) = 5N^2$ and hence $N = 1/\sqrt{5}$.

Solution to Exercise 1.19. Although the motion is one-dimensional, all position states are incompatible with each other: $\langle x|\, x'\rangle = 0$ unless $x = x'$. Therefore there are infinitely many linearly independent states, and the dimension of the Hilbert space is infinite.

Solution to Exercise 1.21. This immediately follows from Ex. 1.7 and 1.16.

Solution to Exercise 1.22. Let $\{|v_i\rangle\}_{i=1}^N$ be an orthonormal basis for $\mathbb{V}$. Then $|a\rangle = \sum_i a_i |v_i\rangle$ and $|b\rangle = \sum_i b_i |v_i\rangle$. Using the result of Ex. 1.14, we write

$$\begin{aligned} \langle a|\, b\rangle &= \sum_{i,j} a_j^* b_i \langle v_j|\, v_i\rangle \\ &= \sum_{i,j} a_j^* b_i \delta_{ji} \\ &= \sum_i a_i^* b_i . \end{aligned}$$

as we desired.

Solution to Exercise 1.23. According to Ex. 1.21, since there are two vectors in each set, it is enough to show that each set is orthonormal. Recalling the definitions, $|\pm\rangle = (|H\rangle \pm |V\rangle)/\sqrt{2}$, we have

$$\langle +|\,+\rangle = \frac{\langle H|\,H\rangle + \langle V|\,H\rangle + \langle H|\,V\rangle + \langle V|\,V\rangle}{2} = \frac{1+0+0+1}{2} = 1.$$

Similarly,

$$\langle +|\,-\rangle = \frac{\langle H|\,H\rangle + \langle V|\,H\rangle - \langle H|\,V\rangle - \langle V|\,V\rangle}{2} = \frac{1+0-0-1}{2} = 0,$$

and, remembering $\langle -|\,+\rangle = \langle +|\,-\rangle^* = 0^* = 0$ it remains to test $\langle -|\,-\rangle$.

$$\langle -|\,-\rangle = \frac{\langle H|\,H\rangle - \langle V|\,H\rangle - \langle H|\,V\rangle + \langle V|\,V\rangle}{2} = \frac{1-0-0+1}{2} = 1.$$

In a similar set of calcualtions [don't forget the complex conjugation when necessary!], we can show that $\langle R|\,R\rangle = 1$, $\langle R|\,L\rangle = \langle L|\,R\rangle = 0$ and $\langle L|\,L\rangle = 1$. Hence $\{|R\rangle, |L\rangle\}$ is also an orthonormal basis.

Solution to Exercise 1.24. We begin with the decomposition

$$|a\rangle = \sum_i a_i\,|v_i\rangle \tag{6.16}$$

where we have assumed that $\{|v_i\rangle\}_{i=1}^N$ is our basis. We take the inner product of both sides of Eq. (6.16) with an arbitrary basis element $|v\rangle_j$ and find, using the basis orthonormality,

$$\langle v_j|\,a\rangle = \langle v_j|\left(\sum_i a_i\,|v_i\rangle\right) = \sum_i a_i\,\langle v_j|\,v_i\rangle = \sum_i a_i\delta_{ji} = a_j.$$

and hence we can also express $|a\rangle$ as $|a\rangle = \sum_i \langle v_i|\,a\rangle\,|v_i\rangle$.

Solution to Exercise 1.25. We will use the result of the previous exercise to find the desired decompositions in the matrix form:

$$\begin{aligned}
\langle +|\,H\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle -|\,H\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle +|\,V\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle -|\,V\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{2}},
\end{aligned}$$

and thus $|H\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$; $|V\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$. Similarly for the circular polarization basis [we perform complex conjugation according to Eq. (1.10)],

$$\begin{aligned}
\langle R|\,H\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle L|\,H\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}};\\
\langle R|\,V\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{i}{\sqrt{2}};\\
\langle L|\,V\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}},
\end{aligned}$$

hence $|H\rangle = (|R\rangle + |L\rangle)/\sqrt{2}$; $|V\rangle = i(-|R\rangle + |L\rangle)/\sqrt{2}$.

Solution to Exercise 1.26. We begin by noting that $|a_i|^2 = |\langle v_i|\, a\rangle|^2$ as we found in Ex. 1.24. Then we have

$$\begin{aligned}
\sum_i |a_i|^2 &= \sum_i |\langle v_i|\, a\rangle|^2 \\
&= \sum_i \langle v_i|\, a\rangle^* \langle v_i|\, a\rangle \\
&= \sum_i \langle a|\, v_i\rangle \langle v_i|\, a\rangle \quad \text{(by definition of the inner product, Property 3)} \\
&= \langle a| \left(\sum_i |v_i\rangle \langle v_i|\, a\rangle\right) \\
&= \langle a|\, a\rangle \quad \text{(according to Eq. (1.12))} \\
&= 1,
\end{aligned}$$

which is what we wanted.

Solution to Exercise 1.27. Using Table 1.1, we express the states $|a\rangle$ and $|b\rangle$ in the canonical basis:

$$\begin{aligned}
|a\rangle &= |30^\circ\rangle = \frac{\sqrt{3}\,|H\rangle + |V\rangle}{2} \leftrightarrow \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \\
|b\rangle &= |{-30^\circ}\rangle = \frac{\sqrt{3}\,|H\rangle - |V\rangle}{2} \leftrightarrow \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix}.
\end{aligned}$$

To decompose these states in the diagonal and circular polarization bases, we use the result of Ex. 1.24. to do so, we need to find the inner products of $|a\rangle$ and $|b\rangle$ with the basis elements. These inner products are convenient to calculate by expressing the vectors in the matrix form, in the canonical basis.

$$\begin{aligned}
\langle +|\, a\rangle &\overset{\text{canonical basis}}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}+1}{2\sqrt{2}}; \\
\langle -|\, a\rangle &\overset{\text{canonical basis}}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}$$

and thus, the decomposition of $|a\rangle$ in the diagonal polarization basis is,

$$|a\rangle \overset{\text{diagonal basis}}{\longleftrightarrow} \begin{pmatrix} \langle +|\, a\rangle \\ \langle -|\, a\rangle \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}+1 \\ \sqrt{3}-1 \end{pmatrix}. \tag{6.17}$$

Similarly, we obtain

$$|b\rangle \overset{\text{diagonal basis}}{\longleftrightarrow} \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}-1 \\ \sqrt{3}+1 \end{pmatrix}. \tag{6.18}$$

For the circular polarization basis,

$$\begin{aligned}
\langle R|\, a\rangle &\overset{\text{canonical basis}}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}-i}{2\sqrt{2}}; \\
\langle L|\, a\rangle &\overset{\text{canonical basis}}{=} \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = \frac{\sqrt{3}+i}{2\sqrt{2}}
\end{aligned}$$

therefore

$$|a\rangle \overset{\text{circular basis}}{\longleftrightarrow} \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}-i\\ \sqrt{3}+i\end{pmatrix} \tag{6.19}$$

and similarly

$$|b\rangle \overset{\text{circular basis}}{\longleftrightarrow} \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{3}+i\\ \sqrt{3}-i\end{pmatrix}. \tag{6.20}$$

To find the inner product in each of the three bases, we use Eq. (1.10).

$$\begin{aligned}
\langle a|\, b\rangle_{\text{canonical basis}} &= \frac{1}{4}\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\begin{pmatrix}\sqrt{3}\\ -1\end{pmatrix} = \frac{1}{2}\\
\langle a|\, b\rangle_{\text{diagonal basis}} &= \frac{1}{8}\begin{pmatrix}\sqrt{3}+1 & \sqrt{3}-1\end{pmatrix}\begin{pmatrix}\sqrt{3}-1\\ \sqrt{3}+1\end{pmatrix} = \frac{1}{2}\\
\langle a|\, b\rangle_{\text{circular basis}} &= \frac{1}{8}\begin{pmatrix}\sqrt{3}+i & \sqrt{3}-i\end{pmatrix}\begin{pmatrix}\sqrt{3}+i\\ \sqrt{3}-i\end{pmatrix} = \frac{1}{2}
\end{aligned}$$

and all three inner products are the same, confirming the theory.

Solution to Exercise 1.28. We first notice that none of the vectors $|v_i\rangle$ can be zero because, according to Eq. (1.14), each of them is a linear combination of linearly independent vectors $|w_1\rangle, \ldots |w_j\rangle$.

Second, we need to verify that the vectors $|v_i\rangle$ are orthogonal to each other. To this end, it is enough to show that each new vector $|v_{k+1}\rangle$ is orthogonal to the previous $|v_j\rangle$ where $j \le k$. We proceed as follows:

$$\begin{aligned}
\langle v_j|\, v_{k+1}\rangle &= \langle v_j| \left(\mathcal{N}\left[|w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|\, w_{k+1}\rangle\, |v_i\rangle\right]\right)\\
&= \mathcal{N}\left[\langle v_j|\, w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|\, w_{k+1}\rangle \langle v_j|\, v_i\rangle\right]\\
&= \mathcal{N}\left[\langle v_j|\, w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|\, w_{k+1}\rangle\, \delta_{ji}\right]\\
&= \mathcal{N}\left[\langle v_j|\, w_{k+1}\rangle - \langle v_j|\, w_{k+1}\rangle\right]\\
&= \mathcal{N}(0),
\end{aligned}$$

hence the set $\{|v_i\rangle\}$ is orthogonal. It is also normalized and contains $N = \dim \mathbb{V}$ elements. According to Ex. 1.21, such a set forms a basis in $\mathbb{V}$.

Solution to Exercise 1.29. To prove the Cauchy-Schwarz inequality, we first note that for any vectors $|a\rangle, |b\rangle$ and complex scalar λ, the relationship $0 \le \|\, |a\rangle - \lambda\, |b\rangle\, \|^2$ holds. Expanding out the norm, we see that

$$\begin{aligned}
0 &\le (\langle a| - \lambda^* \langle b|)\,(|a\rangle - \lambda\, |b\rangle)\\
&= \langle a|\, a\rangle - \lambda \langle a|\, b\rangle - \lambda^* \langle a|\, b\rangle + |\lambda|^2 \langle b|\, b\rangle
\end{aligned}$$

and, taking $\lambda = \langle b|\, a\rangle / \langle b|\, b\rangle = \langle a|\, b\rangle^* / \langle b|\, b\rangle$, the expression turns into the following

$$\begin{aligned} 0 &\leq \langle a|\, a\rangle - \frac{\langle a|\, b\rangle^* \langle a|\, b\rangle}{\langle b|\, b\rangle} - \frac{\langle b|\, a\rangle \langle b|\, a\rangle^*}{\langle b|\, b\rangle} + \left|\frac{\langle b|\, a\rangle}{\langle b|\, b\rangle}\right|^2 \langle b|\, b\rangle \\ &= \langle a|\, a\rangle - 2\frac{|\langle b|\, a\rangle|^2}{\langle b|\, b\rangle} + \frac{|\langle b|\, a\rangle|^2}{\langle b|\, b\rangle} \\ &= \langle a|\, a\rangle - \frac{|\langle b|\, a\rangle|^2}{\langle b|\, b\rangle}, \end{aligned}$$

from which we find

$$|\langle a|\, b\rangle|^2 \leq \langle a|\, a\rangle \langle b|\, b\rangle .$$

Taking the square root of each side yields the required result

$$|\langle a|\, b\rangle| \leq \| |a\rangle \| \times \| |b\rangle \| \tag{6.21}$$

Solution to Exercise 1.30. The triangle inequality is a direct consequence of the Cauchy-Schwarz inequality. To see this, we start by taking the norm of a vector of the form $|a\rangle + |b\rangle$

$$\begin{aligned} \| |a\rangle + |b\rangle \|^2 &= (\langle a| + \langle b|)(|a\rangle + |b\rangle) \\ &= \langle a|\, a\rangle + \langle a|\, b\rangle + \langle b|\, a\rangle + \langle b|\, b\rangle \\ &= \| |a\rangle \|^2 + \| |b\rangle \|^2 + \langle a|\, b\rangle^* + \langle a|\, b\rangle \\ &= \| |a\rangle \|^2 + \| |b\rangle \|^2 + 2\mathrm{Re}\{\langle \mathrm{a}|\, \mathrm{b}\rangle\} \\ &\leq \| |a\rangle \|^2 + \| |b\rangle \|^2 + 2|\langle a|\, b\rangle| \qquad (\mathrm{since\ Re\{z\}} \leq |\mathrm{z}|) \\ &\leq \| |a\rangle \|^2 + \| |b\rangle \|^2 + 2\| |a\rangle \| \times \| |b\rangle \| \qquad (\mathrm{by\ Cauchy-Schwarz}) \\ &= (\| |a\rangle \| + \| |b\rangle \|)^2 . \end{aligned}$$

and, finally, taking the square root of both sides gives us the result we require

$$\| |a\rangle + |b\rangle \| \leq \| |a\rangle \| + \| |b\rangle \|. \tag{6.22}$$

Solution to Exercise 1.31. One $\lambda/4$ waveplate with its optical axis oriented at 45° from horizontal would do the trick. Such a waveplate would "see" the states $|H\rangle$ and $|V\rangle$ as the states $|+\rangle$ and $|-\rangle$, and would accordingly convert $|H\rangle \leftrightarrow |R\rangle$ and $|V\rangle \leftrightarrow |L\rangle$.

Hence with such a waveplate and a polarizing beam splitter, one could put all right circularly polarized photons to one detector and left to another.

Solution to Exercise 1.32. To solve this problem, we use the inner products between the different states that were found previously in exercise 1.27.

a)

$$\begin{aligned} |\langle 30^\circ|\, H\rangle|^2 &= \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4} \\ |\langle 30^\circ|\, V\rangle|^2 &= \left|\frac{1}{2}\right|^2 = \frac{1}{4} \end{aligned}$$

Hence the measurement gives a 75% chance of being in state $|H\rangle$ and a 25% chance of being in state $|V\rangle$.

b)

$$\begin{aligned}
|\langle 30^\circ|\,+\rangle|^2 &= \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{4+2\sqrt{3}}{8} \\
|\langle 30^\circ|\,-\rangle|^2 &= \left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2 = \frac{4-2\sqrt{3}}{8}
\end{aligned}$$

Hence the measurement gives an $\approx 93.3\%$ chance of being in state $|+\rangle$ and an $\approx 6.7\%$ chance of being in state $|-\rangle$.

c)

$$\begin{aligned}
|\langle 30^\circ|\,R\rangle|^2 &= \left|\frac{\sqrt{3}-i}{2\sqrt{2}}\right|^2 = \frac{1}{2} \\
|\langle 30^\circ|\,L\rangle|^2 &= \left|\frac{\sqrt{3}+i}{2\sqrt{2}}\right|^2 = \frac{1}{2}
\end{aligned}$$

Hence the measurement gives a 50% chance of being in state $|+\rangle$ and a 50% chance of being in state $|-\rangle$.

Solution to Exercise 1.33. It benefits us to first consider some things we know about the general polarization state $|\psi\rangle = \alpha\,|H\rangle + \beta\,|V\rangle$.

If we express $\alpha = r_\alpha e^{i\phi_\alpha}$ and $\beta = r_\beta e^{i\phi_\beta}$ (where the ϕ's are real, r's are both real and non-negative) in polar form and recall that a change in the overall phase of the system does not effect its physics, we may multiply the state $|\psi\rangle$ by the phase factor $e^{-i\phi_\beta}$ to see that it can also be expressed as $|\psi\rangle = r_\alpha\,|H\rangle + r_\beta e^{i\phi}\,|V\rangle$ [where we have defined the new variable $\phi \equiv \phi_\beta - \phi_\alpha$]. Finally, we note that the normalization condition on $|\psi\rangle$ tells us that $r_\alpha^2 + |r_\beta e^{i\phi}|^2 = 1 = r_\alpha^2 + r_\beta^2$ and hence that we may define a single new variable θ with $\cos\theta \equiv r_\alpha$ and $\sin\theta \equiv r_\beta$. Because both of the latter quantities are non-negative, we can assume the value of θ to lie within the interval $[0, \pi/2]$.

This has shown us that we may express any state $|\psi\rangle$ in the canonical basis as

$$|\psi\rangle \longleftrightarrow \begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}. \tag{6.23}$$

Our task is to find two unknown variables, θ and ϕ. Measuring in the three bases will give us three independent equations, which is enough information to find the state. We begin by expressing the probability p_H of detecting horizontal polarization in terms of θ and ϕ.

$$p_H = |\langle H|\,\psi\rangle|^2 = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}\right|^2 = |\cos\theta|^2 = \cos^2\theta,$$

from which we find

$$\begin{aligned}
\cos\theta &= \sqrt{p_H} \\
\sin\theta &= \sqrt{1-\cos^2\theta} = \sqrt{1-p_H}
\end{aligned}$$

(because we assumed that $0 < \theta < \pi/2$, we can dismiss the negative solutions for $\cos\theta$ and $\sin\theta$).

We now use these findings to determine ϕ. We write the probability of detecting the +45° polarized state as follows:

$$\begin{aligned}
p_+ &= |\langle +|\ \psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}\right|^2 \\
&= 1/2|\cos\theta + \sin\theta e^{i\phi}|^2 \\
&= 1/2\left[\cos^2\theta + \sin^2\theta + \sin\theta\cos\theta\left(e^{i\phi} + e^{-i\phi}\right)\right] \\
&= 1/2 + 2\sqrt{(1-p_H)(p_H)}\cos\phi \\
&\Rightarrow \quad \cos\phi = \frac{p_+ - 1/2}{2\sqrt{(p_H)(1-p_H)}}.
\end{aligned}$$

Because there are no restrictions on ϕ, we can assume it to vary over the interval $[-\pi, \pi]$. Then the above equation has two solutions:

$$\phi = \pm\cos^{-1}\left(\frac{p_+ - 1/2}{\sqrt{(p_H)(1-p_H)}}\right).$$

We see that even though there are only two unknowns, measurements in the canonical and diagonal bases leave some ambiguity in the state $|\psi\rangle$. For example, such measurements cannot distinguish the right- and left-circular polarized states because both of them have $p_H = p_+ = 1/2$. In order to remove this ambiguity, we use the remaining piece of information, the measurement in the circular polarization basis. We write:

$$\begin{aligned}
p_R &= |\langle R|\ \psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix}\right|^2 \\
&= 1/2|\cos\theta + -i\sin\theta e^{i\phi}|^2 \\
&= 1/2\left[\cos^2\theta + -i\sin^2\theta + \sin\theta\cos\theta\left(e^{i\phi} - e^{-i\phi}\right)\right] \\
&= 1/2 + 2\sqrt{(1-p_H)(p_H)}\sin\phi \\
&\Rightarrow \quad \sin\phi = \frac{p_R - 1/2}{\sqrt{(p_H)(1-p_H)}}.
\end{aligned}$$

Now we have both the sine and cosine, which unambiguously define the value of ϕ.

We have thus expressed the state (equivalently, its parameters θ and ϕ) in terms of the probability values p_H and p_+ that we can find experimentally.

Solution to Exercise 1.34.

a) Assume for simplicity that the space we want to measure states in is two-dimensional. An apparatus that is able to resolve one of the states, say $|a\rangle$, with perfect certainty, must include a projective measurement associated with the basis $\{|a\rangle, |a_\perp\rangle\}$, where $|a_\perp\rangle$ is a vector defined to be orthogonal to $|a\rangle$.

   If this apparatus is now applied to measure $|b\rangle$, according to the Second Postulate it would project onto $|a\rangle$ with a nonzero probability equal to the $|\langle a|\ b\rangle|^2$. Therefore, with some probability, the projective measurement will yield the same outcome for $|a\rangle$ and $|b\rangle$. No matter what classical processing the output of this measurement is subjected to, this indistinguishability will persist.

b) Such a device would be made with two sub-devices, one measuring in the $\{|a\rangle, |a_\perp\rangle\}$ basis, one measuring in the $\{|b\rangle, |b_\perp\rangle\}$ basis, and a randomizer that sends states to one device or the other completely randomly. If the b-measurer detects a $|b_\perp\rangle$ state, one would know the input state was certainly not $|b\rangle$. Similarly, if the a-measurer detects a $|a_\perp\rangle$ state, the input state is certainly not $|a\rangle$. Finally, if the a-measurer detects a $|a\rangle$ or the b-measurer detects a $|b\rangle$, the input state is uncertain by the argument in part (a) of this problem.

Solution to Exercise 1.35. To show that $\mathbb{V}^\dagger$ is linear, we must check the linear space axioms set out in definition (1.1). The following eight axioms are for all $|a\rangle$, $|b\rangle$ and $|c\rangle$ in $\mathbb{V}^\dagger$ and λ, μ in $\mathbb{F}$

I *Commutativity*

$$\langle a| + \langle b| \;=\; \text{Adjoint}\,(|a\rangle + |b\rangle) = \text{Adjoint}\,(|b\rangle + |a\rangle) = \langle b| + \langle a|$$

II *Associativity*

$$\begin{aligned}(\langle a| + \langle b|) + \langle c| &= \text{Adjoint}\,((|a\rangle + |b\rangle) + |c\rangle) = \text{Adjoint}\,(|a\rangle + (|b\rangle + |c\rangle))\\ &= \langle a| + (\langle b| + \langle c|)\end{aligned}$$

III *Zero Element*

We try $\langle \text{zero}|$:

$$\langle a| + \langle \text{zero}| \;=\; \text{Adjoint}\,(|a\rangle + |\text{zero}\rangle) = \text{Adjoint}\,(|a\rangle) = \langle a|$$

hence $\langle \text{zero}|$ is our zero element.

IV *Inverses*

We define $-\langle a| \equiv \text{Adjoint}\,(-|a\rangle)$ and verify that it is the inverse of $\langle a|$:

$$\langle a| + (-\langle a|) \;=\; \text{Adjoint}\,(|a\rangle + -(|a\rangle)) = \text{Adjoint}\,(|\text{zero}\rangle) = \langle \text{zero}|\,.$$

V *Vector Distributivity*

$$\begin{aligned}\lambda(\langle a| + \langle b|) &= \text{Adjoint}\,(\lambda^*(|a\rangle + |b\rangle)) = \text{Adjoint}\,(\lambda^*\,|a\rangle + \lambda * |b\rangle)\\ &= \lambda\,\langle a| + \lambda\,\langle b|\end{aligned}$$

VI *Scalar Distributivity*

$$\begin{aligned}(\lambda + \mu)\,\langle a| &= \text{Adjoint}\,((\lambda + \mu)^*\,|a\rangle) = \text{Adjoint}\,((\lambda^* + \mu^*)\,|a\rangle)\\ &= \text{Adjoint}\,(\lambda^*\,|a\rangle + \mu^*\,|a\rangle) = \lambda\,\langle a| + \mu\,\langle a|\end{aligned}$$

VII *Scalar Associativity*

$$\begin{aligned}\lambda(\mu\,\langle a|) &= \text{Adjoint}\,(\lambda^*(\mu^*\,|a\rangle)) = \text{Adjoint}\,((\lambda^*\mu^*)\,|a\rangle)\\ &= \text{Adjoint}\,((\lambda\mu)^*\,|a\rangle) = (\lambda\mu)\,\langle a|\end{aligned}$$

VIII *Scalar Identity*

$$1 \cdot \langle a| \;=\; \text{Adjoint}\,(1^* \cdot |a\rangle) = \text{Adjoint}\,(1 \cdot |a\rangle) = \text{Adjoint}\,(|a\rangle) = \langle a|$$

Solution to Exercise 1.36. Let $\{|v_i\rangle\}$ be a basis for $\mathbb{V}$. To show that $\{\langle v_i|\}$ is a basis for $\mathbb{V}^\dagger$, we must show that the set is linearly independent and spans the space.

Span. Let $\langle x| \in \mathbb{V}^\dagger$. Then, correspondingly, $|x\rangle \in \mathbb{V}$, and since $\{|v_i\rangle\}$ is a basis,

$$|x\rangle = \sum_i \lambda_i^*\,|v_i\rangle$$

for some set of $\lambda_i \in \mathbb{F}$. Taking the adjoint of both sides gives

$$\begin{aligned}
\langle x| &= \text{Adjoint}\,(|x\rangle) \\
&= \text{Adjoint}\left(\sum_i \lambda_i\,|v_i\rangle\right) \\
&= \sum_i \lambda_i^*\,\langle v_i|
\end{aligned}$$

and hence we see that $\langle x|$ is expressible via the set $\{\langle v_i|\}$. In other words, this set spans $\mathbb{V}^\dagger$.

Independence. Suppose the zero element $\langle\text{zero}|$ can be expressed as a linear combination of the set $\{\langle v_i|\}$, say $\langle\text{zero}| = \sum \lambda_i\,\langle v_i|$. We then see

$$\begin{aligned}
& \langle\text{zero}| = \sum_i \lambda_i\,\langle v_i| \\
\Rightarrow\quad & \text{Adjoint}\,(|\text{zero}\rangle) = \text{Adjoint}\left(\sum_i \lambda_i^*\,|v_i\rangle\right) \\
\Rightarrow\quad & |\text{zero}\rangle = \sum_i \lambda_i^*\,|v_i\rangle \qquad \text{(from Ex. (1.35))},
\end{aligned}$$

so the basis $\{|v_i\rangle\}$ of $\mathbb{V}$ is not linearly independent. We arrive at a contradiction.

Solution to Exercise 1.37. We already know the matrix decomposition for $|R\rangle$ and $|L\rangle$ in the canonical basis as

$$|R\rangle \leftrightarrow 1/\sqrt{2}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |L\rangle \leftrightarrow 1/\sqrt{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and hence, following Eq.(1.20), we see that the decompositions must be

$$\langle R| \leftrightarrow 1/\sqrt{2}\begin{pmatrix} 1 & -i \end{pmatrix} \qquad \langle L| \leftrightarrow 1/\sqrt{2}\begin{pmatrix} 1 & i \end{pmatrix}$$

Solution to Exercise 1.38.

a) $\hat{A}$ is linear since

$$\hat{A}(|a\rangle + |b\rangle) = 0 = 0 + 0 = \hat{A}\,|a\rangle + \hat{A}\,|b\rangle$$

and

$$\hat{A}(\lambda\,|a\rangle) = 0 = \lambda 0 = \lambda\hat{A}\,|a\rangle$$

b) $\hat{\mathbf{1}}$ is linear since

$$\hat{\mathbf{1}}(|a\rangle + |b\rangle) = |a\rangle + |b\rangle = \hat{\mathbf{1}}\,|a\rangle + \hat{\mathbf{1}}\,|b\rangle$$

and

$$\hat{\mathbf{1}}(\lambda\,|a\rangle) = \lambda\,|a\rangle = \lambda\hat{\mathbf{1}}\,|a\rangle$$

c) $\hat{A}$ is linear since

$$\begin{aligned}\hat{A}\left(\left(\begin{array}{c}x\\y\end{array}\right)+\left(\begin{array}{c}x'\\y'\end{array}\right)\right) &= \hat{A}\left(\begin{array}{c}x+x'\\y+y'\end{array}\right)=\left(\begin{array}{c}x+x'\\-y-y'\end{array}\right)\\ &= \left(\left(\begin{array}{c}x\\-y\end{array}\right)+\left(\begin{array}{c}x'\\-y'\end{array}\right)\right)\\ &= \hat{A}\left(\begin{array}{c}x\\y\end{array}\right)+\hat{A}\left(\begin{array}{c}x'\\y'\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}\hat{A}\left(\lambda\left(\begin{array}{c}x\\y\end{array}\right)\right) &= \hat{A}\left(\begin{array}{c}\lambda x\\\lambda y\end{array}\right)=\left(\begin{array}{c}\lambda x\\-\lambda y\end{array}\right)\\ &= \lambda\left(\begin{array}{c}x\\-y\end{array}\right)=\lambda\hat{A}\left(\begin{array}{c}x\\y\end{array}\right)\end{aligned}$$

d) $\hat{A}$ is *not* linear. We can see this by focusing on the y-component of two vectors $\left(\begin{array}{c}x\\y\end{array}\right)$ and $\left(\begin{array}{c}x'\\y'\end{array}\right)$. On one hand we know that

$$\begin{aligned}\hat{A}\left(\left(\begin{array}{c}x\\y\end{array}\right)+\left(\begin{array}{c}x'\\y'\end{array}\right)\right) &= \hat{A}\left(\begin{array}{c}x+x'\\y+y'\end{array}\right)=\left(\begin{array}{c}x+x'+y+y'\\(x+x')(y+y')\end{array}\right)\\ &= \left(\begin{array}{c}x+x'+y+y'\\xy+x'y+xy'+x'y'\end{array}\right)\end{aligned}$$

but on the other we have

$$\begin{aligned}\hat{A}\left(\begin{array}{c}x\\y\end{array}\right)+\hat{A}\left(\begin{array}{c}x'\\y'\end{array}\right) &= \left(\begin{array}{c}x+y\\xy\end{array}\right)+\left(\begin{array}{c}x'+y'\\x'y'\end{array}\right)\\ &= \left(\begin{array}{c}x+y+x'+y'\\xy+x'y'\end{array}\right)\end{aligned}$$

We see that the y-components differ by $x'y+xy'$, and hence the operator cannot be linear.

e) We assume for a contradiction that $\hat{A}$ is linear. Consider then

$$\left(\begin{array}{c}1\\1\end{array}\right)=\hat{A}\left(\begin{array}{c}0\\0\end{array}\right)=\hat{A}\left(\left(\begin{array}{c}0\\0\end{array}\right)+\left(\begin{array}{c}0\\0\end{array}\right)\right)=\hat{A}\left(\begin{array}{c}0\\0\end{array}\right)+\hat{A}\left(\begin{array}{c}0\\0\end{array}\right)=\left(\begin{array}{c}1\\1\end{array}\right)+\left(\begin{array}{c}1\\1\end{array}\right)=\left(\begin{array}{c}2\\2\end{array}\right)$$

Hence $\hat{A}$ must not be linear.

f) This operator is linear. This is easiest to picture geometrically: taking a sum of vectors $\vec{a}$ and $\vec{b}$ each rotated by an angle ϕ is the same as first adding the vectors and equal and rotating the sum. Similarly, rotating then scaling a vector is the same as scaling it then rotating it.

Solution to Exercise 1.39. We can define the zero operator as the operator that maps every vector onto $|\text{zero}\rangle$. For every operator $\hat{A}$, we can define the opposite, $-\hat{A}$, according to

$$(-\hat{A})\,|a\rangle \equiv -(\hat{A}\,|a\rangle) \tag{6.24}$$

Then linear operators comply with all properties introduced in Definition 1.1. We leave the verification of this fact to the reader as an independent exercise.

Solution to Exercise 1.40. Consider vector $(1,0)$. If we rotate it by $\pi/2$, we obtain $(0,1)$, and a subsequent flip around the horizontal axis results in $(0,-1)$. If we perform these operations in the reverse order, the flip will have no effect so the resulting vector is $(0,1)$.

Solution to Exercise 1.41. Let us act with operator $\hat{A}(\hat{B}\hat{C})$ on some vector $\hat{a}$. According to Definition 1.19, we find

$$\hat{A}(\hat{B}\hat{C})\left|a\right\rangle = \hat{A}[(\hat{B}\hat{C})\left|a\right\rangle] = \hat{A}[\hat{B}(\hat{C}\left|a\right\rangle)]. \tag{6.25}$$

In other words, in order to implement the action of operator $\hat{A}(\hat{B}\hat{C})$, we must first apply operator $\hat{C}$ to vector $\left|a\right\rangle$, then apply $\hat{B}$ to the result, then apply $\hat{A}$ to the result.

Now let us look at operator $(\hat{A}\hat{B})\hat{C}$. We have

$$(\hat{A}\hat{B})\hat{C}\left|a\right\rangle = (\hat{A}\hat{B})(\hat{C}\left|a\right\rangle) = \hat{A}[\hat{B}(\hat{C}\left|a\right\rangle)]. \tag{6.26}$$

We see that operators $\hat{A}(\hat{B}\hat{C})$ and $(\hat{A}\hat{B})\hat{C}$ map any vector in the same way, i.e. they are equal operators.

Solution to Exercise 1.42. Note that in the basis $\{\left|v_i\right\rangle\}$, we have the relation $\left\langle v\right|_i \hat{\mathbf{1}} \left|v\right\rangle_j = \left\langle v_i \middle| v_j\right\rangle = \delta_{ij}$. The matrix of the identity operator is just the identity matrix:

$$\hat{\mathbf{1}} \leftrightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Solution to Exercise 1.43. We call the basis $\{\left|v_i\right\rangle\}$. Recalling Eq. (1.11), we find the ith element of the decomposition of $\hat{A}\left|a\right\rangle$ into this basis as

$$\left\langle v_i\right|\hat{A}\left|a\right\rangle = \left\langle v_i\right|\hat{A}\left(\sum_j a_j\left|v_j\right\rangle\right) = \sum_j a_j\left\langle v_i\right|\hat{A}\left|v_j\right\rangle = \sum_j A_{ij}a_j$$

But this expression is the same as that for the ith element of the matrix product

$$\begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} \sum_j A_{1j}a_j \\ \vdots \\ \sum_j A_{Nj}a_j \end{pmatrix}.$$

Solution to Exercise 1.44.

a) *Linearity.* Assuming $\hat{A}$ is linear and testing both linearity conditions on $\lambda\hat{A}$ at once, we have

$$\lambda\hat{A}(\mu_a\left|a\right\rangle + \mu_b\left|b\right\rangle) = \lambda(\mu_a\hat{A}\left|a\right\rangle + \mu_b\hat{A}\left|b\right\rangle) = \lambda\mu_a\hat{A}\left|a\right\rangle + \lambda\mu_b\hat{A}\left|b\right\rangle = \mu_a(\lambda\hat{A}\left|a\right\rangle) + \mu_b(\lambda\hat{A}\left|b\right\rangle)$$

Hence $\lambda\hat{A}$ is linear.

Matrix Representation. Let $\{\left|v_i\right\rangle\}$ be our basis. Then

$$(\lambda\hat{A})_{ij} = \left\langle v_i\right|\lambda\hat{A}\left|v_j\right\rangle = \lambda\left\langle v_i\right|\hat{A}\left|v_j\right\rangle = \lambda A_{ij}$$

b) *Linearity.* Assuming $\hat{A}$ and $\hat{B}$ are linear and recalling the definition of operator addition, we see by testing both linearity conditions at once that

$$\begin{aligned}(\hat{A}+\hat{B})(\mu_a\left|a\right\rangle + \mu_b\left|b\right\rangle) &= \hat{A}(\mu_a\left|a\right\rangle + \mu_b\left|b\right\rangle) + \hat{B}(\mu_a\left|a\right\rangle + \mu_b\left|b\right\rangle) \\ &= \hat{A}\mu_a\left|a\right\rangle + \hat{A}\mu_b\left|b\right\rangle + \hat{B}\mu_a\left|a\right\rangle + \hat{B}\mu_b\left|b\right\rangle \\ &= \mu_a\hat{A}\left|a\right\rangle + \mu_a\hat{B}\left|a\right\rangle + \mu_b\hat{A}\left|b\right\rangle + \mu_b\hat{B}\left|b\right\rangle \\ &= \mu_a(\hat{A}+\hat{B})\left|a\right\rangle + \mu_b(\hat{A}+\hat{B})\left|b\right\rangle\end{aligned}$$

Hence $\hat{A}+\hat{B}$ is linear.

Matrix Representation. Let $\{|v_i\rangle\}$ be our basis.

$$\begin{aligned}(\hat{A}+\hat{B})_{ij} &= \langle v_i|(\hat{A}+\hat{B}|v_j\rangle) = \langle v_i|(\hat{A}|v_j\rangle + \hat{B}|v_j\rangle)\\ &= \langle v_i|\hat{A}|v_j\rangle + \langle v_i|(\hat{B}|v_j\rangle = A_{ij}+B_{ij}\end{aligned}$$

c) *Linearity.* Assuming $\hat{A}$ and $\hat{B}$ are linear and recalling the definition of operator multiplication (definition (1.19)), we see by testing both linearity conditions at once that

$$\begin{aligned}(\hat{A}\hat{B})(\mu_a|a\rangle+\mu_b|b\rangle) &= \hat{A}(\hat{B}(\mu_a|a\rangle+\mu_b|b\rangle))\\ &= \hat{A}(\mu_a\hat{B}|a\rangle+\mu_b\hat{B}|b\rangle)\\ &= \mu_a\hat{A}\hat{B}|a\rangle+\mu_b\hat{A}\hat{B}|b\rangle)\\ &= \mu_a(\hat{A}\hat{B})|a\rangle+\mu_b(\hat{A}\hat{B})|b\rangle)\end{aligned}$$

Hence $\hat{A}\hat{B}$ is linear.

Matrix Representation. Let $\{|v_i\rangle\}$ be our basis. According to Ex. 1.43), we have $\hat{B}|v_j\rangle = \sum_l B_{lj}|v_l\rangle$. Then

$$\begin{aligned}(\hat{A}\hat{B})_{ij} &= \langle v_i|\hat{A}\hat{B}|v_j\rangle = \langle v_i|\hat{A}(\hat{B}|v_j\rangle)\\ &= \langle v_i|\hat{A}\left(\sum_l B_{lj}|v_l\rangle\right) = \sum_l B_{lj}\langle v_i|\hat{A}|v_l\rangle) = \sum_l A_{il}B_{lj}\end{aligned}$$

The final sum in the equation is the defining formula for matrix multiplication. Hence, the matrix of a product of operators is the product of matrices of operators, or, given matrix decompositions

$$\hat{A}\longleftrightarrow\begin{pmatrix}A_{11} & \cdots & A_{1N}\\ \vdots & \ddots & \vdots\\ A_{N1} & \cdots & A_{NN}\end{pmatrix}\quad\text{and}\quad \hat{\mathrm{B}}\longleftrightarrow\begin{pmatrix}B_{11} & \cdots & B_{1N}\\ \vdots & \ddots & \vdots\\ B_{N1} & \cdots & B_{NN}\end{pmatrix}$$

the decomposition of the product will be

$$\hat{A}\hat{B}\longleftrightarrow\begin{pmatrix}A_{11} & \cdots & A_{1N}\\ \vdots & \ddots & \vdots\\ A_{N1} & \cdots & A_{NN}\end{pmatrix}\begin{pmatrix}B_{11} & \cdots & B_{1N}\\ \vdots & \ddots & \vdots\\ B_{N1} & \cdots & B_{NN}\end{pmatrix}\tag{6.27}$$

Solution to Exercise 1.45. We first turn our efforts toward finding the matrix representation of $\hat{R}_\theta$, the operator that rotates vectors through an angle θ in the counterlockwise direction. For our calculation using the standard basis of $\mathbb{R}^2$, $\{\vec{e_x},\vec{e_y}\}$, the unit vectors along the x- and y-axes, which are orthonormal under the standard dot product.

The effect of rotating $\vec{e_x}$ through an angle θ is a new unit vector forming angle θ with the x-axis. This vector thus has projection onto the x-axis $|\vec{e_x}|\cos\theta = \cos\theta$ and y-axis of $|\vec{e_y}|\sin\theta = \sin\theta$ and so we know that $\hat{R}_\theta\vec{e_x} = \cos\theta\ \vec{e_x} + \sin\theta\ \vec{e_y}$. Similarly, rotation of $\vec{e_y}$ through the same angle gives $\hat{R}_\theta = -\sin\theta\ \vec{e_x} + \cos\theta\ \vec{e_y}$.

It remains to find the matrix elements of R_θ.

$$\begin{aligned}
R_{\theta xx} &= \vec{e_x}\cdot(\hat{R}_\theta\vec{e_x}) = \vec{e_x}\cdot(\cos\theta\ \vec{e_x}+\sin\theta\ \vec{e_y})\\
&= \cos\theta\ \vec{e_x}\cdot\vec{e_x}+\sin\theta\ \vec{e_x}\cdot\vec{e_y} = \cos\theta
\end{aligned}$$

$$\begin{aligned}
R_{\theta xy} &= \vec{e_x}\cdot(\hat{R}_\theta\vec{e_y}) = \vec{e_x}\cdot(-\sin\theta\ \vec{e_x}+\cos\theta\ \vec{e_y})\\
&= -\sin\theta\ \vec{e_x}\cdot\vec{e_x}+\cos\theta\ \vec{e_x}\cdot\vec{e_y} = -\sin\theta
\end{aligned}$$

$$\begin{aligned}
R_{\theta yx} &= \vec{e_y}\cdot(\hat{R}_\theta\vec{e_x}) = \vec{e_y}\cdot(\cos\theta\ \vec{e_x}+\sin\theta\ \vec{e_y})\\
&= \cos\theta\ \vec{e_y}\cdot\vec{e_x}+\sin\theta\ \vec{e_y}\cdot\vec{e_y} = \sin\theta
\end{aligned}$$

$$\begin{aligned}
R_{\theta yy} &= \vec{e_y}\cdot(\hat{R}_\theta\vec{e_y}) = \vec{e_y}\cdot(-\sin\theta\ \vec{e_x}+\cos\theta\ \vec{e_y})\\
&= -\sin\theta\ \vec{e_y}\cdot\vec{e_x}+\cos\theta\ \vec{e_y}\cdot\vec{e_y} = \cos\theta
\end{aligned}$$

and we see that

$$\hat{R}_\theta \leftrightarrow \begin{pmatrix} R_{\theta xx} & R_{\theta xy}\\ R_{\theta yx} & R_{\theta yy}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{6.28}$$

We now have the machinery to calculate what the matrix representation of the product of $\hat{A}=\hat{R}_\phi$ and $\hat{B}=\hat{R}_\theta$ is, using the result found in equation (6.27)

$$\begin{aligned}
\hat{A}\hat{B} &= \hat{R}_\phi\hat{R}_\theta \leftrightarrow \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\\
&= \begin{pmatrix}\cos\phi\cos\theta-\sin\phi\sin\theta & -\cos\phi\sin\theta-\sin\phi\cos\theta\\ \sin\phi\cos\theta+\cos\phi\sin\theta & -\sin\phi\sin\theta+\cos\phi\cos\theta\end{pmatrix}\\
&= \begin{pmatrix}\cos(\phi+\theta) & -\sin(\phi+\theta)\\ \sin(\phi+\theta) & \cos(\phi+\theta)\end{pmatrix}.
\end{aligned}$$

As expected, the matrix of $\hat{A}\hat{B}$ is identical to that of the rotation by angle $\theta+\phi$.

Solution to Exercise 1.46. Recall from the solution to exercise 1.44 part c) that for any linear operator $\hat{A}$ in a basis $\{|v_i\rangle\}$, we have that $\hat{A}|v_k\rangle=\sum_j A_{jk}|v_j\rangle$ and that the kets $\{|v_i\rangle\}$ correspond to a column with a one at the ith position and zeroes elsewhere. This gives us matrix representation

$$\hat{A}|v_k\rangle \leftrightarrow A_{1k}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}+A_{2k}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}+\cdots+A_{Nk}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}=\begin{pmatrix}A_{1k}\\A_{2k}\\\vdots\\A_{Nk}\end{pmatrix}$$

Solution to Exercise 1.47.

a) In the canonical basis, we have $|v_1\rangle=|H\rangle$ and $|v_2\rangle=|V\rangle$. We thus know that

$$\begin{aligned}
\hat{A}|v_1\rangle &= |R\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle+i|v_2\rangle);\\
\hat{A}|v_2\rangle &= 2|H\rangle = 2|v_1\rangle.
\end{aligned}$$

Now we find the matrix elements according to

$$\begin{aligned} A_{11} &= \left\langle v_1 \middle| \hat{A} \middle| v_1 \right\rangle = 1/\sqrt{2}; \\ A_{12} &= \left\langle v_1 \middle| \hat{A} \middle| v_2 \right\rangle = 2; \\ A_{21} &= \left\langle v_2 \middle| \hat{A} \middle| v_1 \right\rangle = i/\sqrt{2}; \\ A_{22} &= \left\langle v_2 \middle| \hat{A} \middle| v_2 \right\rangle = 0. \end{aligned}$$

So the matrix is

$$\hat{A} \leftrightarrow \begin{pmatrix} 1/\sqrt{2} & 2 \\ i/\sqrt{2} & 0 \end{pmatrix}. \tag{6.29}$$

b) Here we are not given explicit information on how the operator transforms the canonical basis elements. However, we can calculate this. Since $|H\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ and $|V\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$ and because we are dealing with a linear operator, we have

$$\begin{aligned} \hat{A}\,|H\rangle &= (\hat{A}\,|+\rangle + \hat{A}\,|-\rangle)/\sqrt{2} = (|R\rangle + |H\rangle)/\sqrt{2} = (1 + \frac{1}{\sqrt{2}})\,|H\rangle + \frac{i}{\sqrt{2}}\,|V\rangle\,; \\ \hat{A}\,|V\rangle &= (\hat{A}\,|+\rangle - \hat{A}\,|-\rangle)/\sqrt{2} = (|R\rangle - |H\rangle)/\sqrt{2} = (-1 + \frac{1}{\sqrt{2}})\,|H\rangle + \frac{i}{\sqrt{2}}\,|V\rangle\,. \end{aligned}$$

Now following the footsteps of section (a), we obtain the matrix

$$\hat{A} \leftrightarrow \begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & -1 + \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{6.30}$$

Solution to Exercise 1.48.

a) We know that a $\lambda/2$-waveplate with its optical axis oriented vertically will change the state $|H\rangle$ into the state $|H\rangle$ and the state $|V\rangle$ into the state $-|V\rangle$ from Appendix A. Taking the matrix elements, we see

$$\begin{aligned} \langle H|\,(\widehat{\lambda/2})\,|H\rangle &= \langle H|\,H\rangle = 1 \\ \langle H|\,(\widehat{\lambda/2})\,|V\rangle &= -\langle H|\,V\rangle = 0 \\ \langle V|\,(\widehat{\lambda/2})\,|H\rangle &= \langle V|\,H\rangle = 0 \\ \langle V|\,(\widehat{\lambda/2})\,|V\rangle &= -\langle V|\,V\rangle = -1 \end{aligned}$$

and hence the matrix is

$$\widehat{\lambda/2} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.31}$$

b) A $\lambda/2$ waveplate at 45° will map the canonical basis states as follows:

$$|H\rangle \to |V\rangle \text{ and } |V\rangle \to |H\rangle\,. \tag{6.32}$$

This gives us canonical matrix elements

$$\begin{aligned} \langle H|\,(\widehat{\lambda/2})_{45^\circ}\,|H\rangle &= \langle H|\,V\rangle = 0 \\ \langle H|\,(\widehat{\lambda/2})_{45^\circ}\,|V\rangle &= \langle H|\,H\rangle = 1 \\ \langle V|\,(\widehat{\lambda/2})_{45^\circ}\,|H\rangle &= \langle V|\,V\rangle = 1 \\ \langle V|\,(\widehat{\lambda/2})_{45^\circ}\,|V\rangle &= \langle H|\,V\rangle = 0 \end{aligned}$$

and the matrix is

$$\widehat{\lambda/2}_{45^\circ} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.33}$$

c) A $\lambda/4$ waveplate with the optical axis oriented vertically maps

$$|H\rangle \to |H\rangle \text{ and } |V\rangle \to i\,|V\rangle\,. \tag{6.34}$$

By analogy with part (a) of this problem we find

$$\widehat{\lambda/4} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{6.35}$$

d) Because the optical axis of the waveplate has been rotated, Eq. (6.34) transforms into

$$|30^\circ\rangle \to |30^\circ\rangle \text{ and } |120^\circ\rangle \to i\,|120^\circ\rangle \tag{6.36}$$

To find the matrix of the waveplate operator in the canonical basis, we however need to find the its action upon states $|H\rangle$ and $|V\rangle$. A quick calculation shows that

$$\begin{aligned} |H\rangle &= \frac{\sqrt{3}}{2}\,|30^\circ\rangle - \frac{1}{2}\,|120^\circ\rangle \\ |V\rangle &= \frac{1}{2}\,|30^\circ\rangle + \frac{\sqrt{3}}{2}\,|120^\circ\rangle\,, \end{aligned}$$

from which, using Eq. (6.36), we find that these states transform as follows:

$$\begin{aligned} (\widehat{\lambda/4})_{30^\circ}\,|H\rangle &= \frac{\sqrt{3}}{2}\,|30^\circ\rangle - \frac{i}{2}\,|120^\circ\rangle \\ (\widehat{\lambda/4})_{30^\circ}\,|V\rangle &= \frac{1}{2}\,|30^\circ\rangle + i\frac{\sqrt{3}}{2}\,|120^\circ\rangle\,, \end{aligned}$$

We can express the right-hand side of the above equations in the canonical basis

$$\begin{aligned} (\widehat{\lambda/4})_{30^\circ}\,|H\rangle &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\,|H\rangle + \frac{1}{2}\,|V\rangle\right) - \frac{i}{2}\left(-\frac{1}{2}\,|H\rangle + \frac{\sqrt{3}}{2}\,|V\rangle\right) \\ &= \frac{3+i}{4}\,|H\rangle + \frac{\sqrt{3}-\sqrt{3}i}{4}\,|V\rangle \\ (\widehat{\lambda/4})_{30^\circ}\,|V\rangle &= \frac{1}{2}\left(\frac{\sqrt{3}}{2}\,|H\rangle + \frac{1}{2}\,|V\rangle\right) + i\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\,|H\rangle + \frac{\sqrt{3}}{2}\,|V\rangle\right) \\ &= \frac{\sqrt{3}-\sqrt{3}i}{4}\,|H\rangle + \frac{1+3i}{4}\,|V\rangle\,, \end{aligned}$$

so the matrix is

$$(\widehat{\lambda/4})_{30^\circ} \leftrightarrow \begin{pmatrix} \frac{3+i}{4} & \frac{\sqrt{3}-\sqrt{3}i}{4} \\ \frac{\sqrt{3}-\sqrt{3}i}{4} & \frac{1+3i}{4} \end{pmatrix} \tag{6.37}$$

Solution to Exercise 1.49. We test both linearity properties at once. Let $|x\rangle, |y\rangle \in \mathbb{V}$ and $\lambda, \mu \in \mathbb{F}$. Then

$$\begin{aligned} |a\rangle\langle b|\,(\lambda\,|x\rangle + \mu\,|y\rangle) &= \langle b|\,(\lambda\,|x\rangle + \mu\,|y\rangle)\,|a\rangle \\ &= (\lambda\,\langle b|\,x\rangle + \mu\,\langle b|\,y\rangle)\,|a\rangle \\ &= (\lambda\,\langle b|\,x\rangle)\,|a\rangle + (\mu\,\langle b|\,y\rangle)\,|a\rangle \\ &= \lambda(\langle b|\,x\rangle\,|a\rangle) + \mu(\langle b|\,y\rangle\,|a\rangle) \\ &= \lambda(|a\rangle\langle b|)\,|x\rangle + \mu(|a\rangle\langle b|)\,|y\rangle \end{aligned}$$

hence $|a\rangle\langle b|$ is linear.

Solution to Exercise 1.50. Let $\{|v_i\rangle\}$ be the orthonormal basis in which we seek to find the matrix. Then

$$\langle v_i|\left(|a\rangle\langle b|\right)|v_j\rangle = \langle v_i|\left(\langle b|\, v_j\rangle\right)|a\rangle = \langle b|\, v_j\rangle\,\langle v_i|\, a\rangle = a_i b_j^*$$

according to Eq. (1.11), and hence the matrix of $|a\rangle\langle b|$ is

$$|a\rangle\langle b| \leftrightarrow \begin{pmatrix} a_1 b_1^* & \cdots & a_1 b_N^* \\ \vdots & \ddots & \vdots \\ a_N b_1^* & \cdots & a_n b_n^* \end{pmatrix}.$$

Solution to Exercise 1.51. Using the result of the previous exercise, we write:

$$|+\rangle\langle -| \overset{\text{canonical basis}}{\longleftrightarrow} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

To find the matrix in the $\{|R\rangle, |L\rangle\}$ basis, we could use the same method. However, because the expressions for $|+\rangle$ and $|-\rangle$ in the circular basis are not readily available, we perform a straightforward calciulation

$$\begin{aligned}
\langle R|\left(|+\rangle\langle -|\right)|R\rangle &= \langle R|\, +\rangle\,\langle -|\, R\rangle \\
&= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right] = \frac{-i}{2} \\
\langle R|\left(|+\rangle\langle -|\right)|L\rangle &= \langle R|\, +\rangle\,\langle -|\, L\rangle \\
&= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right] = \frac{1}{2} \\
\langle L|\left(|+\rangle\langle -|\right)|R\rangle &= \langle L|\, +\rangle\,\langle -|\, R\rangle \\
&= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\right] = \frac{1}{2} \\
\langle L|\left(|+\rangle\langle -|\right)|L\rangle &= \langle L|\, +\rangle\,\langle -|\, L\rangle \\
&= \left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right] = \frac{i}{2}
\end{aligned}$$

and hence the matrix is

$$|+\rangle\langle -| \overset{\text{circular basis}}{\longleftrightarrow} \frac{1}{2}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} \tag{6.38}$$

Solution to Exercise 1.52. The (k,l) matrix element of the operator in the right-hand side of Eq. (1.30):

$$\langle v_k|\left(\sum_{i,j} A_{ij}\,|v_i\rangle\langle v_j|\right) = \sum_{i,j} A_{ij}\,\langle v_k|\, v_i\rangle\,\langle v_j|\, v_l\rangle = \sum_{i,j} A_{ij}\delta_{ki}\delta_{jl} = A_{kl}$$

equals the matrix element of the operator $\hat{A}$.

Solution to Exercise 1.53. Referring to Ex. 1.52 and recalling that the canonical ortonormal basis is $\{|H\rangle, |V\rangle\}$, we have that

$$\begin{pmatrix} 1 & -3i \\ 3i & 4 \end{pmatrix} \leftrightarrow 1\,|H\rangle\langle H| + (-3i)\,|H\rangle\langle V| + 3i\,|V\rangle\langle H| + 4\,|V\rangle\langle V|$$

Solution to Exercise 1.54.

a) Referring to Ex. 1.52, we see that the decomposition of this matrix into the canonical basis is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \leftrightarrow \frac{1}{\sqrt{2}}\,|H\rangle\langle H| + |H\rangle\langle V| + |V\rangle\langle H| - |V\rangle\langle V| \tag{6.39}$$

b) We obtain

$$\begin{aligned} \hat{H}\,|H\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftrightarrow |{+45^\circ}\rangle \\ \hat{H}\,|V\rangle &\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \leftrightarrow |{-45^\circ}\rangle \end{aligned} \tag{6.40}$$

c) The operation defined by Eq. (6.40) can be implemented by a $\lambda/2$ waveplate with its optical axis oriented at 22.5°

Solution to Exercise 1.55. Recall the result of Ex. 1.52, that in an orthonormal basis $\{|v_i\rangle\}$ we have $\hat{A} = \sum_{i,j} A_{ij}\,|v_i\rangle\langle v_j|$. Then

$$\begin{aligned} \hat{\mathbf{1}} &= \sum_{i,j} \langle v_i|\,\hat{\mathbf{1}}\,|v_j\rangle\,|v_i\rangle\langle v_j| \\ &= \sum_{i,j} \langle v_i|\,v_j\rangle\,|v_i\rangle\langle v_j| \\ &= \sum_{i,j} \delta_{ij}\,|v_i\rangle\langle v_j| \\ &= \sum_{i} |v_i\rangle\langle v_i| \end{aligned}$$

which is what we were after.

Solution to Exercise 1.56. This follows from Definition 1.21 of the outer product operator

$$\langle a|\,(|b\rangle\langle c|)\,|d\rangle = \langle a|\,(\langle c|\,d\rangle\,|b\rangle) = (\langle a|\,b\rangle)(\langle c|\,d\rangle).$$

Solution to Exercise 1.57. It benefits us to recall (Ex. 1.25) that $|H\rangle = 1/\sqrt{2}(|R\rangle + |L\rangle)$ and $|V\rangle = 1/\sqrt{2}i(|R\rangle - |L\rangle)$. Using the expansion of $\hat{A}$ we found in Ex. 1.53, we substitute in the equivalent Dirac form kets to get the expression in terms of the $\{|R\rangle, |L\rangle\}$ matrix elements.

$$\begin{aligned}
\hat{A} &= 1\,|H\rangle\langle H| - 3i\,|H\rangle\langle V| + 3i\,|V\rangle\langle H| + 4\,|V\rangle\langle V| \\
&= 1\left(\frac{|R\rangle + |L\rangle}{\sqrt{2}}\right)\left(\frac{\langle R| + \langle L|}{\sqrt{2}}\right) - 3i\left(\frac{|R\rangle + |L\rangle}{\sqrt{2}}\right)\left(\frac{\langle R| - \langle L|}{-\sqrt{2}i}\right) \\
&\quad +3i\left(\frac{|R\rangle - |L\rangle}{\sqrt{2}i}\right)\left(\frac{\langle R| + \langle L|}{\sqrt{2}}\right) + 4\left(\frac{|R\rangle - |L\rangle}{\sqrt{2}i}\right)\left(\frac{\langle R| - \langle L|}{-\sqrt{2}i}\right) \\
&= \frac{1}{2}(|R\rangle\langle R| + |R\rangle\langle L| + |L\rangle\langle R| + |L\rangle\langle L|) \\
&\quad +\frac{3}{2}(|R\rangle\langle R| + |R\rangle\langle L| - |L\rangle\langle R| - |L\rangle\langle L|) \\
&\quad +\frac{3}{2}(|R\rangle\langle R| - |R\rangle\langle L| + |L\rangle\langle R| - |L\rangle\langle L|) \\
&\quad +\frac{4}{2}(|R\rangle\langle R| - |R\rangle\langle L| + |L\rangle\langle R| + |L\rangle\langle L|) \\
&= \frac{11}{2}\,|R\rangle\langle R| - \frac{3}{2}\,|R\rangle\langle L| - \frac{3}{2}\,|R\rangle\langle L| - \frac{1}{2}\,|L\rangle\langle L|
\end{aligned}$$

and hence

$$\hat{A} \overset{\text{circular basis}}{\longleftrightarrow} \frac{1}{2}\begin{pmatrix} 11 & -3 \\ -3 & -1 \end{pmatrix}$$

Using the second method, we must first find the inner product matrices.

$$\begin{aligned}
\langle H|\,R\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \\
\langle H|\,L\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \\
\langle V|\,R\rangle &= \begin{pmatrix} 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{i}{\sqrt{2}} \\
\langle V|\,L\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{-i}{\sqrt{2}}
\end{aligned}$$

In this case, the v-basis is the canonical basis and the w the circular. We see, then, that the matrix equation is

$$\begin{aligned}
\hat{A} \overset{\text{circular basis}}{\longleftrightarrow} &\quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 1 & -3i \\ 3i & 4 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 7i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 11 & -3 \\ -3 & -1 \end{pmatrix}
\end{aligned}$$

Solution to Exercise 1.58. Let $\{|v_i\rangle\}$ and $\{|w_j\rangle\}$ be two different bases for a space $\mathbb{V}$. Then the trace of the matrix in one basis [say $|v_i\rangle$ is

$$\mathrm{Tr}(\hat{A}) = \sum_i A_{ii} = \sum_i \langle v_i|\,\hat{A}\,|v_i\rangle \tag{6.41}$$

Inserting identity operators, we see

$$\begin{aligned}
\mathrm{Tr}(\hat{A}) &= \sum_i \langle v_i| \mathbf{\hat{1}}\hat{A}\mathbf{\hat{1}} |v_i\rangle \\
&= \sum_{i,j,k} \langle v_i| w_j\rangle \langle w_j| \hat{A} |w_k\rangle \langle w_k| v_i\rangle \\
&= \sum_{i,j,k} \langle w_k| v_i\rangle \langle v_i| w_j\rangle \langle w_j| \hat{A} |w_k\rangle \\
&= \sum_{j,k} \langle w_k| \left(\sum_i |v_i\rangle\langle v_i| \right) |w_j\rangle \langle w_j| \hat{A} |w_k\rangle \\
&= \sum_{j,k} \langle w_k| \mathbf{\hat{1}} |w_j\rangle \langle w_j| \hat{A} |w_k\rangle \\
&= \sum_{j,k} \delta_{jk} \langle w_j| \hat{A} |w_k\rangle \\
&= \sum_{j} \langle w_j| \hat{A} |w_j\rangle
\end{aligned}$$

hence the trace is basis independent.

Solution to Exercise 1.59.

Applying the definition (1.26) of the matrix of an operator, we find for each matrix element of $\hat{\mathbb{P}}_i$

$$\left(\hat{\mathbb{P}}_i\right)_{kl} = \langle v_k| v_i\rangle \langle v_i| v_l\rangle = \delta_{ki}\delta_{il}.$$

Thus the matrix element with $k = l = i$ equals one, and all others vanish.

Solution to Exercise 1.60. Using the definition provided by Eq. (1.33) we write out the expression for the operator $\hat{X}$ acting on one of its eigenstates

$$\hat{X} |x_j\rangle = \left(\sum_i |x_i\rangle \langle x_i| x_i \right) |x_j\rangle \tag{6.42}$$

It is important to note that we use a different dummy variable for the summation index than we do for the target state. Now, because x_i is a scalar (and thus it commutes) we may pull it out to the front of the expression.

$$\hat{X} |x_j\rangle = \sum_i x_i |x_i\rangle \langle x_i| |x_j\rangle \tag{6.43}$$

According to the definition (1.28) of the outer product, we obtain the inner product $\langle x_i| x_j\rangle$ which gives us δ_{ij} because $|x_i\rangle$ and $|x_j\rangle$ are orthonormal basis vectors. All the terms with $i \neq j$ vanish, leaving us with only the jth term of the sum:

$$\hat{X} |x_j\rangle = x_j |x_j\rangle. \tag{6.44}$$

Solution to Exercise 1.61. Each matrix element X_{jk} of $\hat{X}$ can be found according to Eq. (1.26):

$$\langle x_j| \hat{X} |x_k\rangle = \sum_i \langle x_j| x_i\rangle x_i \langle x_i| x_k\rangle. \tag{6.45}$$

Given that $\langle x_j| x_i\rangle = \delta_{ji}$ and $\langle x_i| x_k\rangle = \delta_{ik}$, we find that the matrix element does not vanish only if $j = k$, i.e. on the matrix diagonal. The only surviving term in the sum is that with $i = j = k$, therefore

$$\hat{X} \leftrightarrow \begin{pmatrix} x_1 & 0 & 0 & . & 0 \\ 0 & x_2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & x_N \end{pmatrix}. \tag{6.46}$$

Solution to Exercise 1.62.

a) We begin by writing the observable operator in the Dirac notation according to the definition 1.33:

$$\hat{\sigma}_z = (1)\left|H\right\rangle\!\left\langle H\right| + (-1)\left|V\right\rangle\!\left\langle V\right|. \tag{6.47}$$

There is more than one way to proceed. We can obtain each matrix element individually from Eq. (1.26):

$$\begin{aligned} (\sigma_z)_{11} &= (\sigma_z)_{HH} = \langle H|\,\hat{\sigma}_z\,|H\rangle = 1 && (6.48) \\ (\sigma_z)_{12} &= (\sigma_z)_{HV} = \langle H|\,\hat{\sigma}_z\,|V\rangle = 0 && (6.49) \\ (\sigma_z)_{21} &= (\sigma_z)_{VH} = \langle V|\,\hat{\sigma}_z\,|H\rangle = 0 && (6.50) \\ (\sigma_z)_{22} &= (\sigma_z)_{VV} = \langle V|\,\hat{\sigma}_z\,|V\rangle = -1 && (6.51) \\ & && (6.52) \end{aligned}$$

Thus,

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.53}$$

The above answer can be obtained simply by comparing Eqs.(1.30) and (6.47).

One can also express each element in Eq. (6.47) in the matrix form and obtain

$$\hat{\sigma}_z = |H\rangle\langle H| - |V\rangle\langle V| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 01 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.54}$$

Similarly, using the result of Ex. 1.25, we find:

b)

$$\hat{\sigma}_x = |+\rangle\langle+| - |-\rangle\langle-| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.55}$$

c)

$$\hat{\sigma}_y = |R\rangle\langle R| - |L\rangle\langle L| = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{6.56}$$

Note the complex conjugation of the matrix elements of the rows ("bras").

Solution to Exercise 1.63. By referring to Ex. 1.48(a) we find that the the matrix representation (in the canonical basis) of a $\lambda/2$ waveplate with its optical axis oriented vertically is the $\hat{\sigma}_z$ operator. This waveplate is all that is necessary to implement the $\hat{\sigma}_z$ operator.

Similarly [Ex. 1.48(b)], a $\lambda/2$ waveplate with its optical axis oriented at a 45^o degree angle from vertical is sufficient to implement the $\hat{\sigma}_x$ operator.

If we have multiple optical elements being applied to the photon polarizations the operator for the system of elements may be found by multiplying the operators of the individual elements together. The $\hat{\sigma}_y$ may be implemented (up to an overall phase factor) using two optical elements: a $\lambda/2$

waveplate with its optical axis oriented vertically and a $\lambda/2$ waveplate with its optical axis oriented horizontally:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\hat{\sigma}_y$$

The required additional phase factor of i can be implemented by a quarter-wavelength optical phase shift applied to both polarizations.

Solution to Exercise 1.64. The apparata with eigenbases $\{|H\rangle, |V\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|R\rangle, |L\rangle\}$ are shown in Fig. 1.1. The only modification that needs to be made to convert these projection measurements into measurements of observables is associating a number with each measurement result. For example, we can connect both detectors of Fig. 1.1(a) to a display so that, when a horizontally polarized photon is detected, the display shows "1", and when a vertically polarized photon is detected, the display shows "-1". According to Defn. 1.11, such an apparatus measures observable $\hat{\sigma}_z$. The solution for the other two apparata is analogous.

Solution to Exercise 1.65. If we have a fair, six-sided die the chance of it landing on any given side will be $\frac{1}{6}$. Thus, $\mathrm{pr}_i = \frac{1}{6}$ for all i. The quantity Q_i is the value displayed on the die. Inserting these values into the equation for the expectation value we obtain:

$$\langle Q\rangle = \sum_{i=1}^{6} \mathrm{pr}_i Q_i = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5 \tag{6.57}$$

Solution to Exercise 1.66. From Eq. (1.35) we may write that the expectation value is given by

$$\langle V\rangle = \sum_{i=1}^{N} \mathrm{pr}_i v_i \tag{6.58}$$

where v_i is the value returned by the measurement and pr_i is the probability to detect $|\psi\rangle$ in the state $|v_i\rangle$. The latter equals

$$\mathrm{pr}_i = |\langle v_i|\psi\rangle|^2 = \langle\psi|v_i\rangle\langle v_i|\psi\rangle \tag{6.59}$$

and

$$\begin{aligned}\langle V\rangle &= \sum_{i=1}^{N} v_i \langle\psi|v_i\rangle\langle v_i|\psi\rangle \\ &= \langle\psi|\left(\sum_{i=1}^{N} v_i |v_i\rangle\langle v_i|\right)|\psi\rangle \qquad (6.60) \\ &= \langle\psi|\hat{V}|\psi\rangle \qquad (6.61)\end{aligned}$$

This last step follows from the definition of $\hat{V}$.

Solution to Exercise 1.67. Utilizing Eq. (1.40), we write for the jth matrix element of $\langle a|\hat{A}$ in the basis $\{|v_i\rangle\}$:

$$\left(\langle a|\hat{A}\right)|v_k\rangle = \sum_{ij} A_{ij}\langle a|v_i\rangle\langle v_j|v_k\rangle = \sum_{ij} A_{ij}\langle a|v_i\rangle\delta_{jk} = \sum_i A_{ik}a_i^*.$$

We would obtain the same result by multiplying the matrix representations of $\langle a|$ and $\hat{A}$:

$$\begin{pmatrix} a_1^* & \dots & a_N^* \end{pmatrix}\begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix} = \begin{pmatrix} \sum\limits_i a_i^* A_{i1} & \dots & \sum\limits_i a_i^* A_{iN} \end{pmatrix}.$$

Solution to Exercise 1.68. We implement the proof in the matrix form. For the left-hand side of Eq. (1.41), we find

$$\left(\langle a|\,\hat{A}\right)|c\rangle = \begin{pmatrix} \sum\limits_i a_i^* A_{i1} & \dots & \sum\limits_i a_i^* A_{iN} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \sum_{ik} A_{ik} a_i^* c_k,$$

and for the right hand side,

$$\langle a| \left(\hat{A}\,|c\rangle\right) = \begin{pmatrix} a_1^* & \dots & a_N^* \end{pmatrix} \begin{pmatrix} \sum\limits_k A_{1k} c_k \\ \vdots \\ \sum\limits_k A_{Nk} c_k \end{pmatrix} = \sum_{ik} A_{ik} a_i^* c_k,$$

i.e. the same expression.

Solution to Exercise 1.69. The matrix of the operator $\hat{A}$ is found from Eq. (1.26) as $A_{ij} = \langle v_i|\,\hat{A}\,|v_j\rangle$. If we denote $|b\rangle = \hat{A}\,|v_j\rangle$, it follows from the definition of the adjoint operator that $\langle b| = \langle v_j|\,\hat{A}^\dagger$. Therefore,

$$A_{ij} = \langle v_i|\,\hat{A}\,|v_j\rangle = \langle v_i|\,b\rangle = \langle b|\,v_i\rangle^* = \langle v_j|\,\hat{A}^\dagger\,|v_i\rangle = (A^\dagger)_{ji},$$

where $(A^\dagger)_{ji}$ is the martix element of the operator $\hat{A}^\dagger$ in the jth row, ith column. We see that the matrix of $\hat{A}^\dagger$ obtains from the matrix of $\hat{A}$ by exchanging the row and column numbers (i.e. transposition) and complex conjugation.

Solution to Exercise 1.70. Double transposition of a matrix, combined with a double complex conjugation of each of its elements, will produce the same matrix.

Solution to Exercise 1.71. Transposing and conjugating each of the matrices (1.34) will produce the same matrix. According to the previous exercise, this indicates that the corresponding Pauli operators are Hermitian.

Solution to Exercise 1.72. Let $|c\rangle\langle b| = \hat{A}$. Acting with this operator on an arbitrary ket $|a\rangle$, we obtain vector

$$\hat{A}\,|a\rangle = \langle b|\,a\rangle\,|c\rangle\,,$$

which we denote as $|d\rangle$. Now acting with operator $|b\rangle\langle c|$ on $\langle a|$ from the right, we find

$$\langle a|\,(|b\rangle\langle c|) = \langle a|\,b\rangle\,\langle c| = \langle b|\,a\rangle^*\,\langle c| = \langle d|$$

according to (1.18). This implies that $|b\rangle\langle c| = \hat{A}^\dagger$.

Solution to Exercise 1.73.

a) Let $\hat{C} = \hat{A} + \hat{B}$. Then for the matrix of $\hat{C}^\dagger$ we have

$$(C^\dagger)_{ij} = C_{ji}^* = A_{ji}^* + B_{ji}^* = (A^\dagger)_{ij} + (B^\dagger)_{ij},$$

where $(A^\dagger)_{ij}$ and $(B^\dagger)_{ij}$ are the matrices of operators $\hat{A}^\dagger$ and $\hat{B}^\dagger$, respectively.

b) Similarly, for the matrix of $\hat{C}^\dagger = (\lambda\hat{A})^\dagger$,

$$(C^\dagger)_{ij} = C_{ji}^* = \lambda^* A_{ji}^* = \lambda^* (A^\dagger)_{ij}.$$

c) The matrix of of the operator $\hat{A}\hat{B}$ is the product of the individual matrices [see Ex. 1.44(c)]:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}.$$

For the adjoint matrix, we have

$$\left[(AB)^\dagger\right]_{ij} = (AB)^*_{ji} = \sum_k A^*_{jk} B^*_{ki}. \tag{6.62}$$

On the other hand, the matrix product of $\hat{B}^\dagger$ and $\hat{A}^\dagger$ equals

$$(B^\dagger A^\dagger)_{ij} = \sum_k (B^\dagger)_{ik} (A^\dagger)_{kj} = \sum_k B^*_{ki} A^*_{jk},$$

which is the same as Eq. (6.62).

Solution to Exercise 1.74. Let $\hat{A}\left|\psi\right\rangle = \left|\chi\right\rangle$. Then $\left\langle\psi\right|\hat{A}^\dagger = \left\langle\chi\right|$ and thus

$$\left\langle\psi\right|\hat{A}^\dagger\left|\phi\right\rangle = \left\langle\chi\right|\left.\phi\right\rangle = \left\langle\phi\right|\left.\chi\right\rangle^* = \left\langle\phi\right|\hat{A}\left|\psi\right\rangle^*.$$

Solution to Exercise 1.75. For a simple counter-example we will use the $\hat{\sigma}_z$ and $\hat{\sigma}_y$ operators, which are both Hermitian:

$$\hat{\sigma}_z\hat{\sigma}_y \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

The resulting matrix is not Hermitian:

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Solution to Exercise 1.76.

a) The matrix of an observable operator $\hat{X}$ is given by Eq. (6.46). Because the eigenvalues of an observable are real (i.e. $x_i^* = x_i$), transposition and complex conjugation of the matrix of $\hat{X}$ will produce the same matrix.

b) Consider a Hermitian operator $\hat{A}$. The characteristic equation of its matrix, according to the fundamental theorem of algebra, has at least one root, so $\hat{A}$ has at least one eigenvalue x_1 and a corresponding eigenvector $\left|x\right\rangle_1$:

$$\hat{A}\left|x_1\right\rangle = x_1\left|x_1\right\rangle.$$

We first notice that because $\hat{A}$ is Hermitian,

$$\left\langle x_1\right|\hat{A}\left|x_1\right\rangle = \left\langle x\right|_1\hat{A}\left|x_1\right\rangle^*$$

according to Eq. (1.46), so the quantity

$$\left\langle x_1\right|\hat{A}\left|x_1\right\rangle = x_1\left\langle x_1\right|\left.x_1\right\rangle = x_1$$

is real.

We proceed by selecting vectors $\left|x_2\right\rangle, \ldots, \left|x_N\right\rangle$ such with in addition of the previously found eigenvector $\left|x_1\right\rangle$ they form an orthonormal basis in our Hilbert space $\mathbb{V}$. Because the basis is orthonormal, we find for the first column of the matrix of $\hat{A}$

$$A_{i1} = \left\langle x_i\right|\hat{A}\left|x_1\right\rangle = x_1\left\langle x_i\right|\left.x_1\right\rangle = 0, \qquad i \neq 1. \tag{6.63a}$$

The first row of the operator has the same property:

$$A_{1i} = \langle x_1 | \hat{A} | x_i \rangle = \langle x_i | \hat{A} | x_1 \rangle^* = 0, \qquad i \neq 1, \tag{6.63b}$$

[in the last equation we have again used Eq. (1.46)]. We conclude that the matrix of $\hat{A}$ in the basis $\{|x_i\rangle\}$ has the form

$$\hat{A} \leftrightarrow \left(\begin{array}{c|ccc} x_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \hat{A}_1 & \\ 0 & & & \end{array}\right),$$

where $\hat{A}_1$ is an $(N-1) \times (N-1)$ matrix. Because of the relations (6.63), the operator associated with this matrix maps the subspace $\mathbb{V}_1 \subset \mathbb{V}$ spanned by the set $\{|x_2\rangle, \dots, |x_N\rangle\}$ onto itself. The above argument can thus be repeated for the operator $\hat{A}_1$ in $\mathbb{V}_1$ to obtain a basis $\{|x'_2\rangle, \dots, |x'_N\rangle\}$ in which $|x'_2\rangle$ is an eigenstate of $\hat{A}_1$, and at the same time an eigenstate of $\hat{A}$. In the basis $\{|x_1\rangle, |x'_2\rangle, \dots, |x'_N\rangle\}$ this operator takes the form

$$\hat{A} \leftrightarrow \left(\begin{array}{cc|ccc} x_1 & 0 & 0 & \cdots & 0 \\ 0 & x'_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & \hat{A}_2 & \\ 0 & 0 & & & \end{array}\right).$$

Repeating this procedure additional $N-3$ times, we fully diagonalize $\hat{A}$ and find a set of eigenvectors $\{|x_i\rangle\}$ that forms an orthonormal basis.

Solution to Exercise 1.77.

We will begin with the $\hat{\sigma}_x$ Pauli matrix. In the $\{|H\rangle, |V\rangle\}$ basis,

$$\hat{\sigma}_x \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

To find the eigenvalues λ of this matrix, we set the determinant of $\left(\hat{\sigma}_x - \lambda\hat{\mathbf{1}}\right)$ equal to zero:

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

By solving for λ, we find that our eigenvalues are $\lambda_{1,2} = \pm 1$.

Now we find the eigenvector $|\phi\rangle \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ associated with each of these eigenvalues. To this end, we solve the equation

$$\left(\hat{\sigma}_x - \lambda\hat{\mathbf{1}}\right)|\phi\rangle = |\text{zero}\rangle.$$

We begin with $\lambda_1 = 1$:

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We find $-\alpha + \beta = 0$ and $\alpha - \beta = 0$, thus, $\alpha = \beta$. Also, we apply the normalization condition $\alpha^2 + \beta^2 = 1$ to determine a normalized eigenvector

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftrightarrow |+\rangle . \tag{6.64}$$

Using the same procedure for the $\lambda_2 = -1$ we obtain

$$|\phi_2\rangle = \frac{-1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftrightarrow |-\rangle . \tag{6.65}$$

Now, we follow the same procedure to calculate the eigenvectors and eigenbasis for the other two Pauli matrices. For $\hat{\sigma}_y \leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, we get $\lambda_{1,2} = \pm 1$ and

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \leftrightarrow |R\rangle ; \qquad |\phi_{-1}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \leftrightarrow |L\rangle . \tag{6.66}$$

The matrix $\hat{\sigma}_z \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is already diagonal, so $\lambda_{1,2} = \pm 1$ and

$$|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow |H\rangle ; \qquad |\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow |V\rangle .$$

These results are in agreement with what was found in Exercise 1.62.

Note that in all three cases the matrix representations for the Pauli operators *in their own eigenbases* consist of the eigenvalues placed along the diagonal elements:

$$\hat{\sigma}_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.67}$$

However, the order of the basis vectors is important. The basis vector associated with the $\lambda = 1$ eigenvalue must be first and the basis vector associated with the $\lambda = -1$ eigenvalue must be second for this representation to be valid. Here the eigenbases for the three Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ are $\{|+\rangle, |-\rangle\}$, $\{|R\rangle, |L\rangle\}$ and $\{|H\rangle, |V\rangle\}$ for $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ respectively.

Solution to Exercise 1.78.

The matrix representation for the rotation operator in two-space is

$$\hat{R_\phi} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{6.68}$$

By transposing this matrix we find that it is not Hermitian. To find its eigenvalues, we solve the characteristic equation:

$$\begin{aligned} \det(\hat{R_\phi} - \lambda\hat{I}) &= \begin{vmatrix} \cos\phi - \lambda & -\sin\phi \\ \sin\phi & \cos\phi - \lambda \end{vmatrix} \\ &= \cos^2\phi - 2\lambda\cos\phi + \lambda^2 + \sin^2\phi \\ &= \lambda^2 - 2\cos\phi\lambda + 1 = 0. \end{aligned} \tag{6.69}$$

Thus, our eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= \cos\phi \pm \sqrt{\cos^2\phi - 1} \\ &= \cos\phi \pm i\sin\phi = e^{\pm i\phi} \end{aligned} \tag{6.70}$$

The eigenvalues are complex; therefore, unless $\phi = 0$ or $\phi = \pi$, the matrix $\hat{R_\phi}$ has no eigenvectors in the two-dimensional geometric space $\mathbb{R}^2$. This is not surprising: when we rotate a vector by an angle other than 0 or π, we will not produce a collinear vector. However, if we consider this matrix in the linear space $\mathbb{C}^2$ over the field of complex numbers, it has two eigenvalues $\lambda_{1,2}$ and two corresponding eigenvectors, which we find in the next step.

We begin with the eigenvalue $\lambda_1 = e^{i\phi}$:

$$\begin{pmatrix} -i\sin\phi & -\sin\phi \\ \sin\phi & -i\sin\phi \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or

$$\alpha\sin\phi - i\beta\sin\phi = 0.$$

Solving this equation with the normalization condition $\alpha^2 + \beta^2 = 1$ we determine the eigenvector

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \leftrightarrow |L\rangle\,. \tag{6.71}$$

Similarly, for the $\lambda_2 = e^{i\phi}$ eigenvalue:

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \leftrightarrow |R\rangle\,. \tag{6.72}$$

This result is not surprising: the circularly polarized state (i.e. such that the trajectory of the tip of the electric field vector is a circle) remains circularly polarized when the reference frame is rotated.

Solution to Exercise 1.79. Left for the reader as an independent exercise.

Solution to Exercise 1.80. Left for the reader as an independent exercise.

Solution to Exercise 1.81.

a) $\frac{1}{2}\left([\hat A,\hat B] + \{\hat A,\hat B\}\right) = \frac{1}{2}\left(\hat A\hat B - \hat B\hat A + \hat A\hat B + \hat B\hat A\right) = \hat A\hat B.$

b) $[\hat A,\hat B] = \hat A\hat B - \hat B\hat A = -(\hat B\hat A - \hat A\hat B) = -[\hat B,\hat A].$

c) $[\hat A,\hat B]^\dagger = (\hat A\hat B - \hat B\hat A)^\dagger \overset{(1.44)}{=} \hat B^\dagger\hat A^\dagger - \hat A^\dagger B^\dagger = \left[\hat B^\dagger,\hat A^\dagger\right].$

d) $[\hat A,\hat B]\hat C + \hat B[\hat A,\hat C] = \hat A\hat B\hat C - \hat B\hat A\hat C + \hat B\hat A\hat C - \hat B\hat C\hat A = \hat A\hat B\hat C - \hat B\hat C\hat A = [\hat A,\hat B\hat C].$

Solution to Exercise 1.82.

a) Using Eq. (1.51) we distribute the adjoint into the commutator:

$$\begin{aligned} \left(i[\hat A,\hat B]\right)^\dagger &= -i\left[\hat B^\dagger,\hat A^\dagger\right] \\ &= -i[\hat B^\dagger,\hat A^\dagger] \\ &= -i[\hat B,\hat A] \qquad \text{[because } \hat A \text{ and } \hat B \text{ are Hermitian]} \\ &= i[\hat A,\hat B], \qquad \text{[according to the previous exercise]} \end{aligned}$$

which shows that $(i[\hat A,\hat B])^\dagger$ is Hermitian.

b) $\{\hat A,\hat B\}^\dagger = (\hat A\hat B)^\dagger + (\hat B\hat A)^\dagger = \hat B^\dagger\hat A^\dagger + \hat A^\dagger\hat B^\dagger \overset{(1.44)}{=} \hat A\hat B + \hat B\hat A = \{\hat A,\hat B\}.$

Solution to Exercise 1.83.

We work out the commutator relations for the Pauli operators by expanding them out into matrix notation in the canonical basis according to Eq. (1.34).

$$\begin{aligned}
[\hat{\sigma}_z, \hat{\sigma}_x] &= \hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z \\
&\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow 2i\sigma_y
\end{aligned} \tag{6.73}$$

Thus, we also know that $[\hat{\sigma}_x, \hat{\sigma}_z] = -2i\sigma_y$.

$$\begin{aligned}
[\hat{\sigma}_z, \hat{\sigma}_y] = -[\hat{\sigma}_y, \hat{\sigma}_z] &= \hat{\sigma}_z\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_z \\
&\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&= -2\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \leftrightarrow 2i\sigma_x
\end{aligned} \tag{6.74}$$

Finally,

$$\begin{aligned}
[\hat{\sigma}_x, \hat{\sigma}_y] = -[\hat{\sigma}_y, \hat{\sigma}_x] &= \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x \\
&\leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\
&= 2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \leftrightarrow 2i\sigma_z
\end{aligned} \tag{6.75}$$

Also $[\hat{\sigma}_x, \hat{\sigma}_x] = [\hat{\sigma}_y, \hat{\sigma}_y] = [\hat{\sigma}_z, \hat{\sigma}_z] = 0$ because any operator commutes with itself.

These relations can be summarized by stating:

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\sigma_k, \tag{6.76}$$

where ϵ is the Levi-Civita symbol given by

$$\epsilon_{ijk} \equiv \left\{ \begin{array}{c} +1 \text{ for } ijk = xyz,\ yzx \text{ or } zxy \\ -1 \text{ for } ijk = xzy,\ yxz \text{ or } zyx \\ 0 \text{ otherwise} \end{array} \right\}. \tag{6.77}$$

Solution to Exercise 1.84.

"$\Longrightarrow$": According to Ex. 1.76, we can present both operators in the form $\hat{A} = \sum_i A_i |v_i\rangle\langle v_i|$ and $\hat{B} = \sum_i B_i |v_i\rangle\langle v_i|$. Then:

$$\hat{A}\hat{B} = \sum_{ij} A_i B_j |v_i\rangle \underbrace{\langle v_i | v_j\rangle}_{\delta_{ij}} \langle v_j| = \sum_i A_i B_i |v_i\rangle\langle v_i| = \hat{B}\hat{A} \tag{6.78}$$

"$\Longleftarrow$": (The proof is done only for the case when one of the operators, e.g. $\hat{A}$, has no degenerate eigenvalues. The proof for the general case is suggested as an independent exercise.) Consider one of $\hat{A}$'s eigenvectors $|v_i\rangle$:

$$\hat{A}\,|v_i\rangle = A_i\,|v_i\rangle \tag{6.79}$$

Multiply both sides by $\hat{B}$ on the left:

$$\underbrace{\hat{B}\hat{A}}_{=\hat{A}\hat{B}}|v_i\rangle = A_i\hat{B}\,|v_i\rangle \tag{6.80}$$

and thus

$$\hat{A}\left(\hat{B}\,|v_i\rangle\right) = A_i\left(\hat{B}\,|v_i\rangle\right),$$

so $\hat{B}\,|v_i\rangle$ must be an eigenstate of $\hat{A}$ with the eigenvalue A_i. But on the other hand, $|v_i\rangle$ is also an eigenstate of $\hat{A}$ with the same eigenvalue. Because eigenvalues are nondegenerate, we conclude that $\hat{B}\,|v_i\rangle$ is proportional to $|v_i\rangle$, i.e. $\hat{B}\,|v_i\rangle = B_i\,|v_i\rangle$

Solution to Exercise 1.85. We expand the right-hand side of Eq. (1.53) to write

$$\left\langle \Delta Q^2 \right\rangle = \sum_i \mathrm{pr}_i Q_i^2 - 2\sum_i \mathrm{pr}_i Q_i \langle Q\rangle + \sum_i \mathrm{pr}_i \langle Q\rangle^2 . \tag{6.81}$$

In the last two terms of the above expression, the quantity $\langle Q\rangle$ is the same for all values of i, so it can be factored out of the sum:

$$\left\langle \Delta Q^2 \right\rangle = \sum_i \mathrm{pr}_i Q_i^2 - 2\langle Q\rangle \sum_i \mathrm{pr}_i Q_i + \langle Q\rangle^2 \sum_i \mathrm{pr}_i . \tag{6.82}$$

Using

$$\sum_i \mathrm{pr}_i Q_i^2 = \left\langle Q^2\right\rangle; \quad \sum_i \mathrm{pr}_i Q_i = \langle Q\rangle; \quad \sum_i \mathrm{pr}_i = 1, \tag{6.83}$$

we obtain

$$\left\langle \Delta Q^2 \right\rangle = \left\langle Q^2\right\rangle - 2\langle Q\rangle\langle Q\rangle + \langle Q\rangle^2 = \left\langle Q^2\right\rangle - \langle Q\rangle^2 . \tag{6.84}$$

Solution to Exercise 1.86. The expectation value of the value on top of the die $\langle Q\rangle = 7/2$ (see Ex. 1.65) and the probability of each event is $\frac{1}{6}$. Applying the definition of the uncertainty, we calculate

$$\begin{aligned}
\left\langle \Delta Q^2 \right\rangle &= \sum_i \mathrm{pr}_i \left(Q_i - \langle Q\rangle\right)^2 \\
&= \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 + \frac{1}{6}(4-3.5)^2 + \frac{1}{6}(5-3.5)^2 + \frac{1}{6}(6-3.5)^2 \\
&= \frac{35}{12}
\end{aligned}$$

We can also solve this problem using the result of the previous exercise:

$$\begin{aligned}
\left\langle \Delta Q^2 \right\rangle &= \left\langle Q^2\right\rangle - \langle Q\rangle^2 = \sum_i \mathrm{pr}_i Q_i^2 - \langle Q\rangle^2 \\
&= \frac{1}{6}1^2 + \frac{1}{6}2^2 + \frac{1}{6}3^2 + \frac{1}{6}4^2 + \frac{1}{6}5^2 + \frac{1}{6}6^2 - \left(\frac{7}{2}\right)^2 \\
&= \frac{35}{12}.
\end{aligned}$$

Solution to Exercise 1.87. Using the spectral decomposition

$$\hat{X} = \sum_i x_i\,|x_i\rangle\langle x_i|$$

we obtain

$$\left\langle \Delta X^2 \right\rangle = \sum_i \mathrm{pr}_i \left(x_i - \langle X \rangle\right)^2 . \tag{6.85}$$

On the other hand, since $\hat{\mathbf{1}} = \sum_i |x_i\rangle\langle x_i|$, we find

$$\left(\hat{X} - \langle X \rangle \hat{\mathbf{1}}\right)^2 = \left\{ \sum_i \left[\left(x_i - \langle X \rangle\right) |x_i\rangle\langle x_i| \right] \right\}^2 = \sum_i \left(x_i - \langle X \rangle\right)^2 |x_i\rangle\langle x_i| ,$$

and thus, for the right-hand side of Eq. (1.55), we obtain

$$\left\langle \psi \right| \left(\hat{X} - \left\langle \hat{X} \right\rangle \hat{\mathbf{1}}\right)^2 \left| \psi \right\rangle = \sum_i \mathrm{pr}_i \left(x_i - \langle X \rangle\right)^2 ,$$

which is the same as Eq. (6.85).

To prove Eq. (1.56), we utilize Eq. (6.85) to argue, in a fashion analogous to the classical case (See Ex. 1.85), that

$$\left\langle \Delta X^2 \right\rangle = \sum_i \mathrm{pr}_i x_i^2 - \langle X \rangle^2 . \tag{6.86}$$

The first term in the expression above is the expectation value of operator $\hat{X}^2$.

Solution to Exercise 1.88. If $|\psi\rangle$ is an eigenstate of the operator $\hat{X}$, we have $\hat{X}|\psi\rangle = x|\psi\rangle$ and $\hat{X}^2|\psi\rangle = x^2|\psi\rangle$. Therefore

$$\left\langle \Delta \hat{X}^2 \right\rangle = \left\langle \psi \right| \hat{X}^2 \left| \psi \right\rangle - \left(\left\langle \psi \right| \hat{X} \left| \psi \right\rangle \right)^2 = x^2 \langle \psi | \psi \rangle - \left(x \langle \psi | \psi \rangle\right)^2 = x^2 - x^2 = 0.$$

Conversely, suppose that the uncertainty of measuring the observable $\hat{X}$ in the state $|\psi\rangle$ vanishes. Denoting $\hat{X}|\psi\rangle \equiv |\phi\rangle$, we write

$$\left\langle \Delta \hat{X}^2 \right\rangle = \left\langle \psi \right| \hat{X}^2 \left| \psi \right\rangle - \left(\left\langle \psi \right| \hat{X} \left| \psi \right\rangle \right)^2 = \langle \phi | \phi \rangle - \langle \psi | \phi \rangle^2 ,$$

where in the last equality we have utilized the fact that $\hat{X}$, as an observable, is Hermitian. By assumption, $\left\langle \Delta \hat{X}^2 \right\rangle = 0$, so we have

$$\langle \phi | \phi \rangle = \langle \psi | \phi \rangle^2 . \tag{6.87}$$

Because the state $|\psi\rangle$ is normalized, we can rewrite Eq. (6.87) as

$$\langle \psi | \psi \rangle \langle \phi | \phi \rangle = \langle \psi | \phi \rangle^2 .$$

We now notice that the above equation exemplifies the Cauchy-Schwartz inequality (1.15). As determined in Ex. 1.29, this inequality becomes an equality (as is in our case) if and only if the states $|\psi\rangle$ and $|\phi\rangle$ are collinear, i.e. $|\phi\rangle = \hat{X}|\psi\rangle = x|\psi\rangle$.

Solution to Exercise 1.89.

$$\begin{aligned}
\left\langle \left[\hat{A}\,\hat{B}\right] \right\rangle &= \left\langle \psi \right| \hat{A}\hat{B} \left| \psi \right\rangle - \left\langle \psi \right| \hat{B}\hat{A} \left| \psi \right\rangle \\
&= \left\langle \psi \right| \hat{A}\hat{B} \left| \psi \right\rangle - \left\langle \psi \right| \hat{B}^\dagger\hat{A}^\dagger \left| \psi \right\rangle \\
&= \left\langle \psi \right| \hat{A}\hat{B} \left| \psi \right\rangle - \left\langle \psi \right| \hat{A}\hat{B} \left| \psi \right\rangle^* \qquad \text{[by Eq. (1.46)]} \\
&= \left\langle \hat{A}\hat{B} \right\rangle - \left\langle \hat{A}\hat{B} \right\rangle^* \\
&= 2i\mathrm{Im}\left\langle \hat{A}\hat{B} \right\rangle \qquad \text{[because } z - z^* = 2i\,\mathrm{Im}z \text{ for any complex } z\text{]}
\end{aligned}$$

Similarly:

$$\begin{aligned}\left\langle\left\{\hat{A}\,\hat{B}\right\}\right\rangle &= \left\langle\psi\right|\hat{A}\hat{B}\left|\psi\right\rangle + \left\langle\psi\right|\hat{A}\hat{B}\left|\psi\right\rangle^* \\ &= \left\langle\hat{A}\hat{B}\right\rangle + \left\langle\hat{A}\hat{B}\right\rangle^* \\ &= 2\mathrm{Re}\left\langle\hat{A}\hat{B}\right\rangle\end{aligned}$$

Finally,

$$\left|\left\langle[\hat{A},\hat{B}]\right\rangle\right|^2 = 4\left(\mathrm{Im}\left\langle\hat{A}\hat{B}\right\rangle\right)^2 \le 4\left(\mathrm{Im}\left\langle\hat{A}\hat{B}\right\rangle\right)^2 + 4\left(\mathrm{Re}\left\langle\hat{A}\hat{B}\right\rangle\right)^2 = 4\left|\left\langle\hat{A}\hat{B}\right\rangle\right|^2. \tag{6.88}$$

Solution to Exercise 1.90. The left-hand side of the Cauchy-Schwarz inequality

$$\langle a|\,a\rangle\,\langle b|\,b\rangle \ge |\langle a|\,b\rangle|^2 \tag{6.89}$$

for $|a\rangle = \hat{A}\,|\psi\rangle$ and $|b\rangle = \hat{B}\,|\psi\rangle$, where $\hat{A}$ and $\hat{B}$ are Hermitian operators takes the form

$$\langle a|\,a\rangle\,\langle b|\,b\rangle = \left\langle\psi\right|\hat{A}^\dagger\hat{A}\left|\psi\right\rangle\left\langle\psi\right|\hat{B}^\dagger\hat{B}\left|\psi\right\rangle = \left\langle\psi\right|\hat{A}^2\left|\psi\right\rangle\left\langle\psi\right|\hat{B}^2\left|\psi\right\rangle. \tag{6.90}$$

Similarly, the right-hand side of Eq. (6.89) becomes

$$|\langle a|\,b\rangle|^2 = \left|\left\langle\psi\right|\hat{A}\hat{B}\left|\psi\right\rangle\right|^2, \tag{6.91}$$

so inequality (6.89) takes the form of Eq. (1.60).

Solution to Exercise 1.92. According to Eq. (1.62), $\left\langle\Delta\hat{A}^2\right\rangle = \langle A^2\rangle$ and $\left\langle\Delta\hat{B}^2\right\rangle = \langle B^2\rangle$ and the uncertainty principle (1.61) takes the form:

$$\left\langle\psi\right|\hat{A}^2\left|\psi\right\rangle\left\langle\psi\right|\hat{B}^2\left|\psi\right\rangle \ge \frac{1}{4}\left|\left\langle\psi\right|[\hat{A},\hat{B}]\left|\psi\right\rangle\right| \tag{6.92}$$

This result obtains immediately from Eqs. (1.60) and (1.59).

Solution to Exercise 1.92. Let us define operators $\hat{\alpha} = \hat{A} - \left\langle\hat{A}\right\rangle$ and $\hat{\beta} = \hat{B} - \left\langle\hat{B}\right\rangle$. The expectation values $\langle\alpha\rangle$ and $\langle\beta\rangle$ vanish, so we can use the "simplified" uncertainty principle (6.92) to write

$$\left\langle\Delta\hat{A}^2\right\rangle\left\langle\Delta\hat{B}^2\right\rangle = \langle\hat{\alpha}^2\rangle\left\langle\hat{\beta}^2\right\rangle \ge \frac{1}{4}\left|\left\langle[\hat{\alpha},\hat{\beta}]\right\rangle\right|^2. \tag{6.93}$$

At the same time,

$$\begin{aligned}\left[\hat{\alpha}\,\hat{\beta}\right] &= \hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha} \\ &= (\hat{A} - \left\langle\hat{A}\right\rangle)(\hat{B} - \left\langle\hat{B}\right\rangle) - (\hat{B} - \left\langle\hat{B}\right\rangle)(\hat{A} - \left\langle\hat{A}\right\rangle) \\ &= \hat{A}\hat{B} - \left\langle\hat{A}\right\rangle\hat{B} - \hat{A}\left\langle\hat{B}\right\rangle + \left\langle\hat{A}\right\rangle\left\langle\hat{B}\right\rangle - \hat{B}\hat{A} + \hat{B}\left\langle\hat{A}\right\rangle\left\langle\hat{B}\right\rangle\hat{A} - \left\langle\hat{B}\right\rangle\left\langle\hat{A}\right\rangle \\ &= \hat{A}\hat{B} - \hat{B}\hat{A} \\ &= \left[\hat{A}\,\hat{B}\right],\end{aligned} \tag{6.94}$$

and therefore

$$\left\langle\Delta\hat{A}^2\right\rangle\left\langle\Delta\hat{B}^2\right\rangle \ge \frac{1}{4}\left|\left\langle\left[\hat{A}\,\hat{B}\right]\right\rangle\right|^2. \tag{6.95}$$

The uncertainty principle would not remain valid if the commutator of $\hat{A}$ and $\hat{B}$ were replaced by the anticommutator or product of these operators because in this case Eq. (6.94) would no longer apply.

Solution to Exercise 1.93.

$$\left\langle \Delta\hat{A}^2\right\rangle\left\langle \Delta\hat{B}^2\right\rangle \geq \frac{1}{4}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|^2 = \frac{1}{4}\left|\langle\epsilon\hat{\mathbf{1}}\rangle\right|^2 = \frac{1}{4}|\epsilon|^2$$

Solution to Exercise 1.94.

$$\langle\hat{\sigma}_x\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0; \tag{6.96}$$

$$\langle\hat{\sigma}_y\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0; \tag{6.97}$$

$$\begin{aligned}\left\langle\Delta\hat{\sigma_x}^2\right\rangle &= \left\langle H\right|\hat{\sigma_x}^2\left|H\right\rangle - \left(\langle H|\hat{\sigma_x}|H\rangle\right)^2 \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^2 = 1;\end{aligned}$$

$$\begin{aligned}\left\langle\Delta\hat{\sigma}_y^2\right\rangle &= \left\langle H\right|\hat{\sigma}_y^2\left|H\right\rangle - \langle H|\hat{\sigma}_y|H\rangle^2 \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]^2 = 1.\end{aligned}$$

On the other hand, from Ex. 1.83 we know that $[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\sigma_z$ so

$$\langle[\hat{\sigma_x}, \hat{\sigma_y}]\rangle = \langle H|\,2i\hat{\sigma}_z|\,H\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2i.$$

The uncertainty principle takes the form

$$\left\langle\Delta\hat{\sigma_x}^2\right\rangle\left\langle\Delta\hat{\sigma_y}^2\right\rangle \geq \frac{1}{4}\left|\langle[\hat{\sigma_x}, \hat{\sigma_y}]\rangle\right|^2.$$

Both sides of the inequality equal 1.

Solution to Exercise 1.95. For any nonzero vector $|a\rangle$ there exists vector $|a_1\rangle = |a\rangle / \|\,|a\rangle\,\|$ of length 1. Operator $\hat{U}$ maps this vector onto $|b_1\rangle = \hat{U}|a_1\rangle$, which also has length 1. Because $\hat{U}$ is unitary, we have $|b\rangle = \hat{U}|a\rangle = \|\,|a\rangle\,\|\,|b_1\rangle$ and accordingly $\langle b|\,b\rangle = \|\,|a\rangle\,\|^2\langle b_1|\,b_1\rangle = \|\,|a\rangle\,\|^2$.

Solution to Exercise 1.96. If an operator preserves the inner product, it will also preserve the norm of a vector because the norm is the square root of the inner product of the vector with itself.

To prove the inverse statement, let us consider arbitrary two vectors, $|a\rangle$ and $|b\rangle$. Then for $|c\rangle = |a\rangle + |b\rangle$ we have

$$\langle c|\,c\rangle = \langle a|\,a\rangle + \langle b|\,b\rangle + \langle a|\,b\rangle + \langle a|\,b\rangle^*. \tag{6.98}$$

At the same time, for $|a'\rangle = \hat{U}|a\rangle$, $|b'\rangle = \hat{U}|b\rangle$ and $|c'\rangle = \hat{U}|c\rangle$, we have

$$\langle c'|\,c'\rangle = \langle a'|\,a'\rangle + \langle b'|\,b'\rangle + \langle a'|\,b'\rangle + \langle a'|\,b'\rangle^*. \tag{6.99}$$

Since $\langle a'|\,a'\rangle = \langle a|\,a\rangle$, $\langle b'|\,b'\rangle = \langle b|\,b\rangle$, $\langle c'|\,c'\rangle = \langle c|\,c\rangle$, we see from Eqs. (6.98) and (6.99) that $\mathrm{Re}\,\langle a'|\,b'\rangle = \mathrm{Re}\,\langle a|\,b\rangle$.

By conducting a similar argument with $|c\rangle = |a\rangle + i\,|b\rangle$, we obtain $\mathrm{Im}\,\langle a'|\,b'\rangle = \mathrm{Im}\,\langle a|\,b\rangle$.

Solution to Exercise 1.97.

a) Since a unitary operator preserves inner products, it maps orthonormal basis $\{|w_i\rangle\}$ onto an orthonormal set, which we denote $\{|v_i\rangle\}$. Now since any orthonormal set of N (where $N = \dim \mathbb{V}$) vectors forms a basis (Ex. 1.21), so does $\{|v_i\rangle\}$.

   Operator $\sum_i |v_i\rangle\langle w_i|$ maps every basis element $|w_i\rangle$ onto $|v_i\rangle$, i.e. it maps basis $\{|w_i\rangle\}$ in the same way as does operator $\hat{U}$. This is a sufficient condition for these two operators to be equal (see Sec. 1.10).

b) For any ket $|a\rangle = \sum_i a_i |w_i\rangle$, we have $|a'\rangle = \hat{U}|a\rangle = \sum_i a_i |v_i\rangle$. Accordingly,

$$\langle a|\, a\rangle = \sum_i |a_i|^2 = \langle a'|\, a'\rangle\,.$$

   We see that operator $\hat{U}$ preserves the norm of $|a\rangle$ and hence it is unitary.

Solution to Exercise 1.98. If operator $\hat{U}$ is unitary, it can be written in the form $\hat{U} = \sum_i |v_i\rangle\langle w_i|$, where $\{|w_i\rangle\}$ and $\{|v_i\rangle\}$ are some orthonormal bases. Accordingly,

$$\hat{U}\hat{U}^\dagger = \sum_{ij} |v_i\rangle\langle w_i|\, w_j\rangle\langle v_j| = \sum_{ij} |v_i\rangle\,\delta_{ij}\,\langle v_j| = \sum_i |v_i\rangle\langle v_j| \overset{(1.31)}{=} \hat{\mathbf{1}}.$$

The argument that $\hat{U}^\dagger\hat{U} = \hat{\mathbf{1}}$ is similar.

Now let us prove that any operator $\hat{U}$ that satisfies $\hat{U}^\dagger\hat{U} = \hat{\mathbf{1}}$ preserves the inner product between two arbitrary vectors $|a\rangle$ and $|b\rangle$. Defining $|a'\rangle = \hat{U}|a\rangle$ and $|b'\rangle = \hat{U}|b\rangle$, we have

$$\langle a'|\, b'\rangle = \left\langle a\right|\hat{U}^\dagger\hat{U}\left|b\right\rangle = \langle a|\, b\rangle\,.$$

Solution to Exercise 1.99.

a) If a unitary operator $\hat{U}$ has an eigenvalue x with an associated eigenstate $|x\rangle$, we have for $|x'\rangle = \hat{U}|x\rangle = x|x\rangle$:

$$\langle x'|\, x'\rangle = x^* x\,\langle x|\, x\rangle\,.$$

   Because a unitary operator preserves the inner product, we must have $x^*x = |x|^2 = 1$. This can be satisfied by any $x = e^{i\theta}$ with $\theta \in \mathbb{R}$.

b) If an operator $\hat{U}$ is diagonalizable, its matrix in its eigenbasis has the form

$$\hat{U} \leftrightarrow \begin{pmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N \end{pmatrix}, \tag{6.100}$$

   where u_i are the eigenvalues of absolute value 1 (i.e. such that $u_i^* u_i = 1$). The adjoint matrix is then

$$\hat{U}^\dagger \leftrightarrow \begin{pmatrix} u_1^* & 0 & \cdots & 0 \\ 0 & u_2^* & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N^* \end{pmatrix} \tag{6.101}$$

   and their product is

$$\hat{U}^\dagger U = UU^\dagger \leftrightarrow \begin{pmatrix} u_1^*u_1 & 0 & \cdots & 0 \\ 0 & u_2^*u_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_N^*u_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \leftrightarrow \hat{\mathbf{1}}. \tag{6.102}$$

   This shows that the operator $\hat{U}$ is unitary.

Solution to Exercise 1.100.

a) For the Pauli operators:

$$\begin{aligned}
\sigma_x^\dagger\sigma_x &\leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \hat{\mathbf{1}}; \\
\sigma_y^\dagger\sigma_y &\leftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \hat{\mathbf{1}}; \\
\sigma_z^\dagger\sigma_z &\leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \hat{\mathbf{1}}.
\end{aligned}$$

So all three Pauli operators are unitary.

b) For the rotation operator:

$$\begin{aligned}
\hat{R}_\phi\hat{R}_\phi^\dagger &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}^\dagger \\
&= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \cos\phi\sin\phi & \sin^2\phi + \cos^2\phi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \hat{\mathbf{1}},
\end{aligned}$$

so this operator is unitary, too. This can be understood intuitively: when vectors are rotated, their scalar products do not change.

Solution to Exercise 1.101. The first step is to diagonalize $\hat{A}$. The characterisic equation for this matrix is:

$$\det(\hat{A} - \lambda\hat{1}) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0,$$

from which we find our eigenvalues $\lambda_{1,2} = \{4, -2\}$. The normalized eigenvector associated with the first eigenvalue is

$$|\phi_1\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

and with the second

$$|\phi_2\rangle \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle.$$

This means that our operator can be written as

$$\hat{A} = 4\,|\phi_1\rangle\langle\phi_1| - 2\,|\phi_2\rangle\langle\phi_2|.$$

Now we apply Eq. (1.65) to express $\sqrt{\hat{A}}$ as

$$\begin{aligned}
\sqrt{\hat{A}} &= \sqrt{4}\,|\phi_1\rangle\langle\phi_1| + \sqrt{-2}\,|\phi_2\rangle\langle\phi_2| \\
&= 2\,|\phi_1\rangle\langle\phi_1| + i\sqrt{2}\,|\phi_2\rangle\langle\phi_2| \\
&\leftrightarrow 2\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} + i\sqrt{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} (1+\frac{i}{\sqrt{2}}) & (1-\frac{i}{\sqrt{2}}) \\ (1-\frac{i}{\sqrt{2}}) & (1+\frac{i}{\sqrt{2}}) \end{pmatrix},
\end{aligned}$$

where all matrices are in the canonical basis.

To determine $\ln\hat{A}$, we need to find the logarithm of its eigenvalues, one of which (λ_2) is negative. Logarithm of negative numbers is not defined in the real number space. In the complex number space, we can use the fact that $e^{(2m+1)i\pi} = -1$ (where m is an arbitrary integer) and thus $e^{(2m+1)i\pi+\ln 2} = (-1)\times 2 = -2$. Hence, $\ln(-2) = (2m+1)i\pi + \ln 2$.[1] Thus,

$$\begin{aligned}
\ln\hat{A} &= \ln 4\,|\phi_1\rangle\langle\phi_1| + \ln(-2)\,|\phi_2\rangle\langle\phi_2| \\
&= \frac{1}{2}\begin{pmatrix} \ln 4 + \ln 2 + (2m+1)i\pi & \ln 4 - \ln 2 - (2m+1)i\pi \\ \ln 4 - \ln 2 - (2m+1)i\pi & \ln 4 + \ln 2 + (2m+1)i\pi \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \ln 8 + (2m+1)i\pi & \ln 8 - (2m+1)i\pi \\ \ln 8 - (2m+1)i\pi & \ln 8 + (2m+1)i\pi \end{pmatrix}.
\end{aligned}$$

Solution to Exercise 1.102. The operator $\hat{\sigma}_z \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is already diagonalized in the canonical basis, so

$$e^{i\theta\hat{\sigma}_z} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

.

Recalling that $\hat{\sigma}_x = |+\rangle\langle+| - |-\rangle\langle-|$ (see Ex. 1.62,1.77), we write

$$\begin{aligned}
e^{i\theta\hat{\sigma}_x} &= e^{i\theta}\,|+\rangle\langle+| - e^{-i\theta}\,|-\rangle\langle-| \\
&\leftrightarrow i\theta\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} + e^{-i\theta}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{i\theta} + e^{-i\theta} & e^{i\theta} - e^{-i\theta} \\ e^{i\theta} - e^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
e^{i\theta\hat{\sigma}_y} &= e^{i\theta}\,|R\rangle\langle R| + e^{-i\theta}\,|L\rangle\langle L| \\
&\leftrightarrow i\theta\begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix} + e^{-i\theta}\begin{pmatrix} 1 \\ -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{i\theta} + e^{-i\theta} & -ie^{i\theta} + ie^{-i\theta} \\ ie^{i\theta} - ie^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.
\end{aligned}$$

The answers to this exercise can also be found as particular cases of Ex. 1.107.

Solution to Exercise 1.103. The eigenvalues of $\hat{A}$ are $a_1 = 0$ and $a_2 = 1$ with corresponding eigenstates $|a_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $|a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore

$$e^{i\theta}\hat{A} = e^0\,|a_1\rangle\langle a_1| + e^{i\theta}\,|a_2\rangle\langle a_2| = \frac{1}{2}\begin{pmatrix} e^{i\theta} + 1 & e^{i\theta} - 1 \\ e^{i\theta} - 1 & e^{i\theta} + 1 \end{pmatrix}.$$

Solution to Exercise 1.104.

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[1] Logarithm is an example of *multivalued* functions that are common in complex analysis.

a) The matrices of $\hat{A}$ and $f(\hat{A})$ in the eigenbasis of $\hat{A}$ are

$$\hat{A} \leftrightarrow \begin{pmatrix} a_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_N \end{pmatrix}; \qquad f(\hat{A}) \leftrightarrow \begin{pmatrix} f(a_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & f(a_N) \end{pmatrix}$$

(with a_i being the eigenvalues) and therefore

$$\hat{A}f(\hat{A}) = f(\hat{A})\hat{A} \leftrightarrow \begin{pmatrix} a_1 f(a_1) & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & a_N f(a_N) \end{pmatrix}.$$

Hence $[\hat{A}, f(\hat{A})] = \hat{A}f(\hat{A}) - f(\hat{A})\hat{A} = 0$.

b) With $\{|a_i\rangle\}$ being the eigenbasis of $\hat{A}$ and $\{a_i\}$ the set of corresponding eigenstates,

$$\begin{aligned} \hat{A}^m \hat{A}^n &= \sum_i \left(a_i^m |a_i\rangle\langle a_i|\right)\left(a_j^n |a_j\rangle\langle a_j|\right) \\ &= \sum_{i,j} a_i^m a_j^n |a_i\rangle \langle a_i| a_j\rangle |a_j\rangle \\ &= \sum_{i,j} a_i^m a_j^n \delta_{ij} |a_i\rangle\langle a_j| \\ &= \sum_i a_i^{m+n} |a_i\rangle\langle a_i| \\ &= \hat{A}^{m+n}. \end{aligned}$$

Solution to Exercise 1.105. We can use Eq. (1.65) to expand the function of the operator into the different order sums of terms and then collapse each set of similarly ordered terms back into distinct operators:

$$\begin{aligned} f(\hat{A}) &= \sum_i f(a_i) |a_i\rangle\langle a_i| \\ &= \sum_i \left(f_0 + f_1 a_i + f_2 {a_i}^2 + \cdots\right) |a_i\rangle\langle a_i| \\ &= f_0 \sum_i |a_i\rangle\langle a_i| + f_1 \sum_i a_i |a_i\rangle\langle a_i| + f_2 \sum_i {a_i}^2 |a_i\rangle\langle a_i| + \cdots \\ f(\hat{A}) &= f_0 \hat{1} + f_1 \hat{A} + f_2 \hat{A}^2 + \cdots . \end{aligned}$$

Solution to Exercise 1.106. Any Hermitian operator may be diagonalized with real eigenvalues a_i (see Ex.1.76):

$$\hat{A} = \sum_i a_i |a_i\rangle\langle a_i|.$$

The exponent of this operator,

$$e^{i\hat{A}} = \sum_i e^{ia_i} |a_i\rangle\langle a_i|,$$

has the same eigenstates, but eigenvalues e^{ia_i}. Because all a_i are real, all e^{ia_i} have absolute values equal to 1, so $e^{i\hat{A}}$ is unitary according to Ex. 1.99.

At the same time, $e^{-i\hat{A}} = \sum\limits_i e^{-ia_i}\left|a_i\right\rangle\!\left\langle a_i\right|$, so

$$e^{i\hat{A}}e^{-i\hat{A}} = \sum_i e^{ia_i}e^{-ia_i}\left|a_i\right\rangle\!\left\langle a_i\right| = \sum_i \left|a_i\right\rangle\!\left\langle a_i\right| = \hat{\mathbf{1}}.$$

Solution to Exercise 1.107. In the canonical basis, the operator $\vec{v}\hat{\vec{\sigma}}$ has the following matrix:

$$\begin{aligned}
\vec{v}\hat{\vec{\sigma}} &\leftrightarrow v_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + v_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}.
\end{aligned}$$

This matrix is Hermitian; hence (Ex. 1.76) $\vec{v}\hat{\vec{\sigma}}$ has two eigenvalues $\lambda_{1,2}$ and two associated orthogonal eigenstates $\left|\lambda_{1,2}\right\rangle$. The eigenvalues of $\vec{v}\hat{\vec{\sigma}}$ are found by solving the charateristic equation:

$$\begin{aligned}
\det(\hat{\vec{v}}\sigma - \lambda\hat{1}) &= \begin{vmatrix} v_z - \lambda & v_x - iv_y \\ v_x + iv_y & -v_z - \lambda \end{vmatrix} \\
&= -(v_z - \lambda)(v_z + \lambda) - (v_x - iv_y)(v_x + iv_y) \\
&= \lambda^2 - v_x^2 - v_y^2 - v_z^2 \\
&= \lambda^2 - |\vec{v}|^2 = 0.
\end{aligned}$$

Because $\vec{v}$ is a unit length vector, the eigenvalues are $\lambda_{1,2} = \pm 1$ and thus

$$\vec{v}\hat{\vec{\sigma}} = \left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| - \left|\lambda_2\right\rangle\!\left\langle\lambda_2\right|. \tag{6.103}$$

We can now write the exponent of the operator as:

$$\begin{aligned}
e^{i\theta\vec{v}\hat{\vec{\sigma}}} &= e^{i\theta}\left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| + e^{-i\theta}\left|\lambda_2\right\rangle\!\left\langle\lambda_2\right| \\
&= (\cos\theta + i\sin\theta)\left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| + (\cos\theta - i\sin\theta)\left|\lambda_2\right\rangle\!\left\langle\lambda_2\right| \\
&= \cos\theta(\left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| + \left|\lambda_2\right\rangle\!\left\langle\lambda_2\right|) + i\sin\theta(\left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| - \left|\lambda_2\right\rangle\!\left\langle\lambda_2\right|). \qquad (6.104)
\end{aligned}$$

Although we have not found explicit expressions for $\left|\lambda_1\right\rangle$ and $\left|\lambda_2\right\rangle$, we know from Eq. (1.55) that $\left|\lambda_1\right\rangle\!\left\langle\lambda_1\right| + \left|\lambda_2\right\rangle\!\left\langle\lambda_2\right| = \hat{\mathbf{1}}$. Using this and Eq. (6.103), we can rewrite Eq. (6.104) as

$$e^{i\theta\vec{v}\hat{\vec{\sigma}}}\cos\theta\hat{1} + i\sin\theta\vec{v}\hat{\vec{\sigma}}.$$

Solution to Exercise 1.108. If $\left|\psi(t)\right\rangle = \sum_i \psi_i(t)\left|v_i\right\rangle$, where $\{\left|v_i\right\rangle\}$ is an orthonormal basis, then, because the basis elements are constant in time, and because the Hilbert space is linear,

$$\frac{\mathrm{d}\left|\psi\right\rangle}{\mathrm{d}t} = \lim_{\Delta t\to 0}\sum_i \frac{\psi_i(t+\Delta t) - \psi_i(t)}{\Delta t}\left|v_i\right\rangle = \sum_i \frac{\mathrm{d}\psi_i}{\mathrm{d}t}\left|v_i\right\rangle.$$

By a similar token, the derivative of an operator with matrix $(Y_{ij}(t))$ is matrix $(\mathrm{d}Y_{ij}(t)/\mathrm{d}t)$.

Solution to Exercise 1.109. In the orthonormal basis $\{\left|a_i\right\rangle\}$ that diagonalizes $\hat{A}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e^{i\hat{A}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \left(e^{ia_it}\right)\left|a_i\right\rangle\!\left\langle a_i\right| \\
&= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{ia_it}\right)\left|a_i\right\rangle\!\left\langle a_i\right| \qquad (6.105) \\
&= \sum_i ia_ie^{ia_it}\left|a_i\right\rangle\!\left\langle a_i\right|. \qquad (6.106)
\end{aligned}$$

On the other hand,

$$\begin{aligned} i\hat{A}e^{i\hat{A}t} &= i\sum_i a_i\,|a_i\rangle\langle a_i|\sum_j e^{ia_jt}\,|a_j\rangle\langle a_j| \\ &= i\sum_{i,j} a_ie^{ia_jt}\,|a_i\rangle\,\langle a_i|\,a_j\rangle\,\langle a_j| \\ &= i\sum_i a_ie^{ia_it}\,|a_i\rangle\,\langle a_i|\,, \end{aligned}$$

which is the same as the right-hand side of Eq. (6.106).

Solution to Exercise 1.110.

a) Since $\hat{A}\hat{B} = \hat{B}\hat{A} + c$, we have

$$\begin{aligned} \hat{A}\hat{B}^n &= \hat{A}\underbrace{\hat{B}\hat{B}\hat{B}\ldots\hat{B}}_{n\text{ times}} \\ &= (\hat{B}\hat{A}+c)\underbrace{\hat{B}\hat{B}\ldots\hat{B}}_{n-1\text{ times}} \\ &= \hat{B}\hat{A}\underbrace{\hat{B}\hat{B}\ldots\hat{B}}_{n-1\text{ times}}+c\hat{B}^{n-1} \\ &= \hat{B}\hat{B}\hat{A}\ \underbrace{\hat{B}\ldots\hat{B}}_{n-2\text{ times}}\ +2c\hat{B}^{n-1} \\ &= \ldots = \underbrace{\hat{B}\hat{B}\hat{B}\ldots\hat{B}}_{n\text{ times}}\hat{A}+nc\hat{B}^{n-1} \\ &= \hat{B}^n\hat{A}+nc\hat{B}^{n-1}. \end{aligned} \tag{6.107}$$

Therefore

$$[\hat{A},\hat{B}^n] = \hat{A}\hat{B}^n - \hat{B}^n\hat{A} = nc\hat{B}^{n-1}. \tag{6.108}$$

b) We use the Taylor decomposition of the exponential function of the operator to write

$$\begin{aligned} [\hat{A},e^{\hat{B}}] &= \sum_{n=0}^{\infty}\frac{1}{n!}[\hat{A},\hat{B}^n] \\ &\overset{\text{part (a)}}{=} \sum_{n=0}^{\infty}\frac{1}{n!}nc\hat{B}^{n-1} \\ &= \sum_{n=1}^{\infty}\frac{1}{(n-1)!}c\hat{B}^{n-1} \\ &\overset{n'=n-1}{=} \sum_{n'=0}^{\infty}\frac{1}{(n')!}c\hat{B}^{n'} \\ &= c\hat{e}^{\hat{B}}. \end{aligned} \tag{6.109}$$

c) We begin by using the result of Ex. 1.109 and writing

$$\frac{\mathrm{d}\hat{G}(\lambda)}{\mathrm{d}\lambda} = \hat{A}e^{\lambda\hat{A}}e^{\lambda\hat{B}} + e^{\lambda\hat{A}}\hat{B}e^{\lambda\hat{B}}.$$

In order to bring the above result to the form of the right-hand side of Eq. (1.73), we need to move both $\hat{A}$ and $\hat{B}$ to the right of the exponentials. Each operator commutes with the exponential of itself (Ex. 1.104), but in order to commute operators $\hat{A}$ and $e^{\lambda\hat{B}}$, the result of part (b) must be used. We have

$$\frac{\mathrm{d}\hat{G}(\lambda)}{\mathrm{d}\lambda}=e^{\lambda\hat{A}}e^{\lambda\hat{B}}(\hat{A}+\lambda c)+e^{\lambda\hat{A}}e^{\lambda\hat{B}}\hat{B}=\hat{G}(\lambda)(\hat{A}+\hat{B}+\lambda c).$$

d) By taking the derivative of both sides of Eq. (1.74), we obtain Eq. (1.73). In order to validate the 'boundary condition", we also check the validity of Eq. (1.74) for $\lambda=0$. In this case, both sides of Eq. (1.74) become the identity operator, so the equation holds.

e) For $\lambda=1$, Eq. (1.74) becomes

$$e^{\hat{A}}e^{\hat{B}}=e^{\hat{A}+\hat{B}+c/2}. \tag{6.110}$$

Because c is a number, this equation is equivalent to the Baker-Hausdorff-Campbell formula.

Solution to Exercise 1.111. According to the result of Ex. 1.109:

$$\frac{d}{dt}e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle=-i\frac{\hat{H}}{\hbar}e^{-i\frac{\hat{H}}{\hbar}t}\left|\psi(t=0)\right\rangle=-i\frac{\hat{H}}{\hbar}\left|\psi(t)\right\rangle,$$

which is consistent with the Schrödinger equation (1.76).

The Hamiltonian operator $\hat{H}$ corresponds to a physical observable, energy, and is thus Hermitian. The Schrödinger evolution operator, $e^{-i\frac{\hat{H}}{\hbar}t}$ must then be unitary according to Ex. 1.106.

Solution to Exercise 1.112.

a) By applying Eq. (1.77), as well as the result of Ex. 1.102 to the Hamiltonian we get that the Schrödinger evolution is given by

$$\begin{aligned}\left|\psi(t)\right\rangle &= e^{-i\omega\hat{\sigma}_z t}\left|\psi(t=0)\right\rangle\\ &= \left(e^{-i\omega t}\left|H\right\rangle\!\left\langle H\right|+e^{i\omega t}\left|V\right\rangle\!\left\langle V\right|\right)\left|\psi(t=0)\right\rangle.\end{aligned}$$

The time evolution for a state initially in $\left|\psi(0)\right\rangle=\left|H\right\rangle$ is:

$$\begin{aligned}\left|\psi(t)\right\rangle &= \left(e^{-i\omega t}\left|H\right\rangle\!\left\langle H\right|+e^{i\omega t}\left|V\right\rangle\!\left\langle V\right|\right)\left|H\right\rangle\\ &= e^{-i\omega t}\left|H\right\rangle.\end{aligned}$$

The time evolution for a state initially in $\left|\psi(0)\right\rangle=\left|+\right\rangle$ is:

$$\begin{aligned}\left|\psi(t)\right\rangle &= \left(e^{-i\omega t}\left|H\right\rangle\!\left\langle H\right|+e^{i\omega t}\left|V\right\rangle\!\left\langle V\right|\right)\frac{1}{\sqrt{2}}\left(\left|H\right\rangle+\left|V\right\rangle\right)\\ &= \frac{1}{\sqrt{2}}\left(e^{-i\omega t}\left|H\right\rangle+e^{i\omega t}\left|V\right\rangle\right).\end{aligned}$$

Let us now apply the other technique: solve the differential equation for the state. Let

$$\left|\psi(t)\right\rangle=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \tag{6.111}$$

in the canonical basis. The matrix of the Hamiltonian takes the form

$$\hat{H}=\hbar\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and the Schrödinger equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=-i\omega\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -i\omega \begin{pmatrix} x(t) \\ -y(t) \end{pmatrix}.$$

This expression means that the same differential equation must hold for each row of the matrices in the left- and right-hand side, so we can rewrite it as a system of ordinary differential equations:

$$\begin{cases} \dot{x}(t) = -i\omega x(t) \\ \dot{y}(t) = i\omega y(t) \end{cases}$$

which has the following solution:

$$\begin{cases} x(t) = Ae^{-i\omega t} \\ x(t) = Be^{i\omega t} \end{cases}$$

The coefficients A and B are obtained from the initial conditions. If the initial state is $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then we have $A = 1, B = 0$ and thus

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\omega t} \\ 0 \end{pmatrix} = e^{-i\omega t}\,|H\rangle\,.$$

If the initial state is $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then we have $A = B = \frac{1}{\sqrt{2}}$ and thus

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ e^{i\omega t} \end{pmatrix}.$$

b) We now do the same for σ_x (*cf.* Ex. 1.102):

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\omega\hat{\sigma}_x t}\,|\psi(0)\rangle \\ &= \left[e^{-i\omega t}\,|+\rangle\langle+| + e^{i\omega t}\,|-\rangle\langle-|\right]|\psi(0)\rangle\,. \end{aligned}$$

The time evolution for a state initially in $|\psi(0)\rangle = |H\rangle$ is

$$\begin{aligned} |\psi(t)\rangle &= \left[e^{-i\omega t}\,|+\rangle\langle+| + e^{i\omega t}\,|-\rangle\langle-|\right]|H\rangle \\ &= \frac{1}{\sqrt{2}}(e^{-i\omega t}\,|+\rangle + e^{i\omega t}\,|-\rangle) \\ &= \frac{1}{2}\left[e^{-i\omega t}(|H\rangle + |V\rangle) + e^{i\omega t}(|H\rangle - |V\rangle)\right] \\ &= \cos\omega t\,|H\rangle + i\sin\omega t\,|V\rangle\,. \end{aligned}$$

The time evolution for a state initially in $|\psi(0)\rangle = |+\rangle$ is:

$$\begin{aligned} |\psi(t)\rangle &= \left(e^{-i\omega t}\,|+\rangle\langle+| + e^{i\omega t}\,|-\rangle\langle-|\right)|+\rangle \\ &= e^{-i\omega t}\,|+\rangle\,. \end{aligned}$$

In order to apply the differential equation technique, we again decompose $|\psi(t)\rangle$ according to Eq. (6.111). The matrix of the Hamiltonian takes the form

$$\hat{H} = \hbar\omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the Schrödinger equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -i\omega \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}.$$

The system of equations for the state components is

$$\left\{ \begin{array}{l} \dot{x}(t) = -i\omega y(t) \\ \dot{y}(t) = -i\omega x(t) \end{array} \right. .$$

In order to solve this system we can, for example, take the derivative of both sides of the first equation and substitute $\dot{y}(t)$ from the second one:

$$\dot{x}(t) = -i\omega \dot{y}(t) = -\omega^2 x(t).$$

This equation has solution

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

and, accordingly,

$$y(t) = \dot{x}(t)/(-i\omega) = -Ae^{i\omega t} + Be^{-i\omega t}.$$

For the initial state $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have $A = B = 1/2$ and thus

$$|\psi(t)\rangle = \frac{1}{2} \begin{pmatrix} e^{i\omega t} + e^{-i\omega t} \\ -e^{i\omega t} + e^{-i\omega t} \end{pmatrix} = \begin{pmatrix} \cos\omega t \\ -i\sin\omega t \end{pmatrix}.$$

For the initial state $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we have $A = \frac{1}{\sqrt{2}}, B = 0$ and thus

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\omega t} \\ e^{-i\omega t} \end{pmatrix} = e^{-i\omega t} \left|+\right\rangle .$$

c) Left for the reader as an independent exercise.