Chapter 7

Solutions to chapter 2 problems

Solution to Exercise 2.1. Let us choose an arbitrary $|a\rangle \in \mathbb{V}$ and consider the sum $|\Psi\rangle = |a\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}} + |\text{zero}\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}}$. According to Rule 2 of the tensor product space algebra, we find $|\Psi\rangle = (|a\rangle_{\mathbb{V}} + |\text{zero}\rangle_{\mathbb{V}}) \otimes |b\rangle_{\mathbb{W}} = |a\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}}$. In other words, adding $|\text{zero}\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}}$ to an element of $\mathbb{V} \otimes \mathbb{W}$ did not change this element. According to Ex. 1.2(b), we find that $|\text{zero}\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}}$ must be $|\text{zero}\rangle$.

The second identity is proven in a similar fashion.

Solution to Exercise 2.2. For simplicity, let us consider the polarization Hilbert space of two photons and show that $B = \{|H\rangle \otimes |H\rangle, |H\rangle \otimes |V\rangle, |V\rangle \otimes |H\rangle, |V\rangle \otimes |V\rangle\}$ is a basis.

First, we prove that B is a spanning set. Consider an arbitrary separable vector $|a\rangle \otimes |b\rangle$ of $\mathbb{V} \otimes \mathbb{W}$. Decomposing $|a\rangle$ and $|b\rangle$ into the canonical bases of their home Hilbert spaces,

$$|a\rangle = \alpha |H\rangle + \beta |V\rangle, \text{ and}$$

$$|b\rangle = \gamma |H\rangle + \delta |V\rangle,$$

we use Properties 1 and 2 of the tensor product space to write

$$|a\rangle \otimes |b\rangle = \alpha\gamma |H\rangle \otimes |H\rangle + \alpha\delta |H\rangle \otimes |V\rangle + \beta\gamma |V\rangle \otimes |H\rangle + \beta\delta |V\rangle \otimes |V\rangle. \tag{7.1}$$

In other words, any separable element of $\mathbb{V} \otimes \mathbb{W}$ can be written as a linear combination of elements of B. This property is readily generalized to entangled vectors because any entangled vector is a linear combination of separable vectors.

Second, we need to prove that B is linearly independent. This follows from the fact that all elements of B are orthogonal to each other (see Rule 3 of Defn. 2.1) and the fact that any set of mutually orthogonal vectors is linearly independent (Ex. 1.16).

Solution to Exercise 2.3.

a) $5 \times 2 + (-6i) \times 3 + (-7i) \times 4 = 10 - 46i.$

b) $(2\langle H| + i\langle V|)(2i|H\rangle - 3i|V\rangle) \times (i\langle H| - \langle V|)(|H\rangle + |V\rangle)$
$= [2 \times (2i) + (-i) \times (-3i)][(-i) \times (1) + (-1) \times (1)] = [-3 + 4i][-1 - i] = 7 - i.$

Solution to Exercise 2.4. All three sets form bases, all but the last one are orthonormal. The proof is left for the reader as an independent exercise.

Solution to Exercise 2.5. This is a separable state because the states of Alice's and Bob's photons can be described independently of each other: $|30^\circ\rangle = \sqrt{3}/2\,|H\rangle + 1/2\,|V\rangle$; $|R\rangle = 1/\sqrt{2}\,|H\rangle + i/\sqrt{2}\,|V\rangle$, therefore

$$|30^\circ\rangle \otimes |R\rangle = \frac{\sqrt{3}}{2\sqrt{2}}|HH\rangle + \frac{\sqrt{3}i}{2\sqrt{2}}|HV\rangle + \frac{1}{2\sqrt{2}}|VH\rangle + \frac{i}{2\sqrt{2}}|VV\rangle \leftrightarrow \begin{pmatrix} \sqrt{3}/2\sqrt{2} \\ \sqrt{3}i/2\sqrt{2} \\ 1/2\sqrt{2} \\ i/2\sqrt{2} \end{pmatrix}.$$

Solution to Exercise 2.6. Consider, for example, $|\Phi^+\rangle$. Suppose this state can be written as a product

$$\left|\Psi^+\right\rangle = |a\rangle_{\mathbb{V}} \otimes |b\rangle_{\mathbb{W}}, \tag{7.2}$$

where $|a\rangle$ and $|b\rangle$ are some states in $\mathbb{V}$ and $\mathbb{W}$, respectively. These states can be decomposed into the canonical bases of their respective spaces:

$$|a\rangle = \alpha\,|H\rangle + \beta\,|V\rangle\,, \text{ and}$$

$$|b\rangle = \gamma\,|H\rangle + \delta\,|V\rangle\,.$$

Substituting these decompositions into Eq. (7.2), comparing the result with the definition (2.4) of $|\Phi^+\rangle$, and using the uniqueness of the decomposition of a vector into a basis, we find:

$$\left[\begin{array}{r} \alpha\gamma = 1/\sqrt{2} \\ \alpha\delta = 0 \\ \beta\gamma = 0 \\ \beta\delta = 1/\sqrt{2}. \end{array}\right. \tag{7.3}$$

From the second equation in the above system, we find that either $\alpha = 0$ or $\delta = 0$. From the third equation, $\beta = 0$ or $\gamma = 0$. Therefore, either $\alpha\gamma$ or $\beta\delta$ must vanish, which contradicts the first and fourth equations of system (7.3).

The proof for other Bell states runs similarly.

Solution to Exercise 2.7. The Bell states form a spanning set because the four canonical basis elements can be expressed through these states:

$$|HH\rangle = \left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)/\sqrt{2}; \tag{7.4a}$$
$$|VV\rangle = \left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)/\sqrt{2}; \tag{7.4b}$$
$$|HV\rangle = \left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)/\sqrt{2}; \tag{7.4c}$$
$$|VH\rangle = \left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right)/\sqrt{2}. \tag{7.4d}$$

Because the dimension of this tensor product space is 4, and according to Ex. 1.7(b), the four Bell states form a basis. The orthonormality of this basis can be verified by direct calculation, i.e.:

$$\begin{aligned} \left\langle\Phi^+\middle|\,\Phi^+\right\rangle &= \left(\langle HH|\,HH\rangle + \langle HH|\,VV\rangle + \langle VV|\,HH\rangle + \langle VV|\,VV\rangle\right)/2 = (1+0+0+1)/2 = 1; \\ \left\langle\Phi^+\middle|\,\Phi^-\right\rangle &= \left(\langle HH|\,HH\rangle - \langle HH|\,VV\rangle + \langle VV|\,HH\rangle - \langle VV|\,VV\rangle\right)/2 = (1+0+0-1)/2 = 0 \end{aligned}$$

and so on.

Solution to Exercise 2.8. Knowing that $|H\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$, $|V\rangle = (|+\rangle - |-\rangle)/\sqrt{2}$, we write

$$\left|\Phi^\pm\right\rangle = \frac{|HH\rangle \pm |VV\rangle}{\sqrt{2}} = \frac{|{+}{+}\rangle + |{+}{-}\rangle + |{-}{+}\rangle + |{-}{-}\rangle}{2\sqrt{2}} \pm \frac{|{+}{+}\rangle - |{+}{-}\rangle - |{-}{+}\rangle + |{-}{-}\rangle}{2\sqrt{2}}$$

and thus

$$\left|\Phi^+\right\rangle = \frac{|{+}{+}\rangle + |{-}{-}\rangle}{\sqrt{2}}; \quad \left|\Phi^-\right\rangle = \frac{|{+}{-}\rangle + |{-}{+}\rangle}{\sqrt{2}}.$$

We see that one of the two Bell states that are symmetric in the canonical basis is antisymmetric in the diagonal basis. A measurement in this basis will thus immediately distinguish them.

Solution to Exercise 2.9. For each matrix element of $\hat{A} \otimes \hat{B}$, we can write

$$\begin{aligned} (\hat{A} \otimes \hat{B})_{iji'j'} &= \langle v_i w_j|\,\hat{A} \otimes \hat{B}\,|v_i' w_j'\rangle = [\langle v_i| \otimes \langle w_j|]\left[\left(\hat{A}\,|v_i'\rangle\right) \otimes \left(\hat{B}\,|w_j'\rangle\right)\right] \\ &= \left(\langle v_i|\,\hat{A}\,|v_i'\rangle\right)\left(\langle w_j|\,\hat{B}\,|w_j'\rangle\right) = A_{ii'}B_{jj'}. \end{aligned}$$

In the second equality above, we used the definition of the tensor product operator.

Solution to Exercise 2.10. We choose a random separable state $|ab\rangle \in \mathbb{V} \otimes \mathbb{W}$ and apply the definition of the operator tensor product:

$$\begin{aligned}\hat{A}_1\hat{A}_2 \otimes \hat{B}_1\hat{B}_2 |a\rangle |b\rangle &= \hat{A}_1\hat{A}_2 |a\rangle \otimes \hat{B}_1\hat{B}_2 |b\rangle = \hat{A}_1 \left(\hat{A}_2 |a\rangle\right) \otimes \hat{B}_1 \left(\hat{B}_2 |b\rangle\right) \\ &= \left(\hat{A}_1 \otimes \hat{B}_1\right) \left(\hat{A}_2 |a\rangle \otimes \hat{B}_2 |b\rangle\right) = \left(\hat{A}_1 \otimes \hat{B}_1\right) \left(\hat{A}_2 \otimes \hat{B}_2\right) |a\rangle |b\rangle .\end{aligned}$$

We see that the operators $\hat{A}_1\hat{A}_2 \otimes \hat{B}_1\hat{B}_2$ and $\left(\hat{A}_1 \otimes \hat{B}_1\right) \left(\hat{A}_2 \otimes \hat{B}_2\right)$ act on each separable state in the same way. Extending this derivation to entangled states is straightforward.

Solution to Exercise 2.11. By definition, if a tensor product operator $\hat{A} \otimes \hat{B}$ act on a separable state $|ab\rangle$, it will generate the state $\hat{A}|a\rangle \otimes \hat{B}|b\rangle$. Its adjoint (see Defn. 1.28) must therefore satisfy

$$\text{Adjoint}\left(\hat{A}|a\rangle \otimes \hat{B}|b\rangle\right) = \text{Adjoint}\left[(\hat{A} \otimes \hat{B})|ab\rangle\right] = \langle ab| \left(\hat{A} \otimes \hat{B}\right)^\dagger . \tag{7.5}$$

But, according to Defn. 2.5, the adjoint of the tensor product state is the state

$$\text{Adjoint}\left(\hat{A}|a\rangle \otimes \hat{B}|b\rangle\right) = \text{Adjoint}(\hat{A}|a\rangle) \otimes \text{Adjoint}(\hat{B}|b\rangle) = \langle a| \hat{A}^\dagger \otimes \langle b| \hat{B}^\dagger. \tag{7.6}$$

Comparing the last two equation, we obtain the required identity.

Solution to Exercise 2.12.

a) If the operators $\hat{A}$ in $\mathbb{V}$ and $\hat{B}$ in $\mathbb{W}$ are Hermitean, their matrices satisfy $A_{ii'} = A^*_{i'i}$ and $B_{ii'} = B^*_{i'i}$. Then, according to the result of Ex. 2.9,

$$(\hat{A} \otimes \hat{B})^*_{i'ij'i'} = A^*_{i'i}B^*_{j'j} = A_{ii'}B_{jj'} = (\hat{A} \otimes \hat{B})_{iji'j'}.$$

When one transposes and conjugates the matrix of $(\hat{A} \otimes \hat{B})$, one obtains the same matrix, which is the signature of a Hermitean operator (Ex. 1.69).

b) If operator $\hat{A}$ in $\mathbb{V}$ is unitary, it maps an orthonormal basis $\{|v_i\rangle\}$ onto another orthonormal basis $\{|v'_i\rangle\}$ (see Ex. 1.97). Similarly, a unitary operator $\hat{B}$ in $\mathbb{W}$ transforms between orthonormal bases $\{|w_i\rangle\}$ and $\{|w'_i\rangle\}$. The tensor product of $\hat{A}$ and $\hat{B}$ transforms between $\{|v_iw_j\rangle\}$ and $\{|v'_iw'_j\rangle\}$, which are both orthonormal bases. An operator with such a property must be unitary.

Solution to Exercise 2.13.

a) The simplest way to find the expectation value of $\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B}$ is to write it in the matrix form, which we obtain from Eq. (2.7). We write for the expectation value

$$\left\langle \Psi^- \right| \hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B} \left| \Psi^- \right\rangle \leftrightarrow \frac{1}{2} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

The uncertainty can be found via Eq. (1.54). One could perform a full matrix calculation again, but perhaps it is easier to notice that the square of any Pauli matrix is the identity operator, and thus

$$\left\langle \Psi^- \right| (\hat{\sigma}_{x_A} \otimes \hat{\sigma}_{y_B})^2 \left| \Psi^- \right\rangle = \left\langle \Psi^- \right| \hat{\mathbf{1}} \left| \Psi^- \right\rangle = \left\langle \Psi^- \middle| \Psi^- \right\rangle = 1.$$

The mean square uncertainty thus equals $1 - 0 = 1$.

b) The probability of detecting the state $|R\rangle_A |{-30°}\rangle_B$ is the square absolute value of the overlap

$$\begin{aligned}\text{pr} &= |\left\langle \Psi^- \middle| R_A, -30^\circ_B \right\rangle|^2 = \frac{1}{2} |\langle H_A V_B | R_A, -30°\rangle_B - \langle V_A H_B | R_A, -30°\rangle_B|^2 \\ &= \frac{1}{2} |\langle H| R\rangle_A \langle V| -30°\rangle_B - \langle V| R\rangle_A \langle H| -30°\rangle_B|^2 = \frac{1}{2} \left| \frac{1}{\sqrt{2}} \frac{-1}{2} - \frac{i}{\sqrt{2}} \frac{\sqrt{3}}{2} \right|^2 = \frac{1}{4}.\end{aligned}$$

c) Similarly,

$$\mathrm{pr} \quad = \quad \left| \frac{1}{3\sqrt{2}} (\langle HV| - \langle VH|)(|HV\rangle + 2\,|VH\rangle + 2\,|VV\rangle) \right|^2 = \frac{1}{18}|1-2|^2 = \frac{1}{18}.$$

Solution to Exercise 2.14.

a) From the description of the operator, we immediately write

$$\widehat{\mathrm{C-NOT}} = |HH\rangle\langle HH| + |HV\rangle\langle HV| + |VV\rangle\langle VH| + |VV\rangle\langle VH| \leftrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \tag{7.7}$$

b) Similarly,

$$\widehat{\mathrm{C-PHASE}} = |HH\rangle\langle HH| + |HV\rangle\langle HV| + |VH\rangle\langle VH| - |VV\rangle\langle VV| \leftrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \tag{7.8}$$

c) The Hadamard gate in the local space maps $|H\rangle \to |+\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$ and $|V\rangle \to |-\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$. In the composed space, the Bob's local Hadamard operator maps

$$|HH\rangle \to |H+\rangle\,;$$
$$|HV\rangle \to |H-\rangle\,;$$
$$|VH\rangle \to |V+\rangle\,;$$
$$|VV\rangle \to |V-\rangle$$

and can thus be written as

$$\hat{\mathbf{1}} \otimes \hat{H} = |H+\rangle\langle HH| + |H-\rangle\langle HV| + |V+\rangle\langle VH| + |V-\rangle\langle VV|\,.$$

Now using Eq. (1.26) we find, in the canonical basis,

$$\hat{\mathbf{1}} \otimes \hat{H} \leftrightarrow \begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & 1 & -1 \end{pmatrix}. \tag{7.9}$$

All these operators are unitary (we can verify this against the definition of unitarity or simply notice that each of them maps an orthonormal basis onto an orthonormal basis). This means one can implement them in a physical process.

Solution to Exercise 2.15. Multiplying matrix (7.9) by (7.8) and again by (7.9), we obtain matrix (7.7).

Solution to Exercise 2.16.

a) Acting with a tensor product of local operators $\hat{A} \otimes \hat{B}$ on a separable state $|a\rangle \otimes |b\rangle$, we obtain, by Definition 2.4, another separable state $(\hat{A}\,|a\rangle) \otimes (\hat{B}\,|b\rangle)$. To show that one cannot "disentangle" by means of local operations, we notice that the inverse of a tensor product of local operators $(\hat{A} \otimes \hat{B})^{-1}$ is $\hat{A}^{-1} \otimes \hat{B}^{-1}$, which is also a tensor product of local operators. If $\hat{A}\otimes\hat{B}$ could "disentangle", the operator $(\hat{A}\otimes\hat{B})^{-1}$ would be able to entangle, which contradicts our first finding.

b) Acting with the c-not gate (7.7) upon a separable state $(|H\rangle + |V\rangle) \otimes |H\rangle / \sqrt{2}$, we obtain $(|HH\rangle + |VV\rangle)/\sqrt{2} = |\Phi^+\rangle$, which, as we know (Ex. 2.6), is entangled. The fact that c-phase can also create entanglement follows from the fact that it can be expressed as a product of local operations (which cannot entangle or "disentangle") and the c-not gate.

Solution to Exercise 2.17.

$$\begin{aligned}
\langle \psi_B | \, \Omega\rangle &= (2\langle H| - i\langle V|)_B(2|HH\rangle + 3|HV\rangle + 4|VH\rangle)_{AB}\\
&= 2|H\rangle_A (2\langle H| - i\langle V|)_B |H\rangle_B + 3|H\rangle_A (2\langle H| - i\langle V|)_B |V\rangle_B + 4|V\rangle_A (2\langle H| - i\langle V|)_B |H\rangle_B\\
&= (4|H\rangle - 3i|H\rangle + 8|V\rangle)_A = [(4-3i)|H\rangle + 8|V\rangle]_A
\end{aligned}$$

$$\begin{aligned}
\langle \Pi | \, \psi_A\rangle &= (2\langle H| - i\langle V|)_A \otimes (-i\langle H| - \langle V|)_B(2|H\rangle + i|V\rangle)_A\\
&= [(2\langle H| - i\langle V|)(2|H\rangle + i|V\rangle)]_A (-i\langle H| - \langle V|)_B\\
&= 5(-i\langle H| - \langle V|)_B
\end{aligned}$$

Mind the complex conjugation when determining the adjoint vectors.

Solution to Exercise 2.18. Let $|\Psi\rangle = \sum_i \lambda_i |a_i\rangle \otimes |b_i\rangle$. Then, according to the definition (2.13a),

$$\langle b| \langle a| \, \Psi\rangle = \langle b| \left(\sum_i \lambda_i \langle a| \, a_i\rangle |b_i\rangle\right) = \sum_i \lambda_i \langle a| \, a_i\rangle \langle b| \, b_i\rangle\,;$$

$$\langle a| \langle b| \, \Psi\rangle = \langle a| \left(\sum_i \lambda_i |a_i\rangle \langle b| \, b_i\rangle\right) = \sum_i \lambda_i \langle a| \, a_i\rangle \langle b| \, b_i\rangle\,;$$

$$\langle ab| \, \Psi\rangle = \langle ab| \left(\sum_i \lambda_i |a_i b_i\rangle\right) = \sum_i \lambda_i \langle a| \, a_i\rangle \langle b| \, b_i\rangle\,,$$

where the last equation is obtained from the definition of the inner product in the tensor product space.

Solution to Exercise 2.19. Let the first element in Alice's orthonormal basis be given by $|v_1\rangle = a|H\rangle + b|V\rangle$, where a and b are arbitrary complex numbers such that $|a|^2 + |b|^2 = 1$. Then

$$\langle v_1 | \, \Psi^+\rangle = \frac{1}{\sqrt{2}} \langle a|H\rangle + b|V\rangle \,|\, |HH\rangle + |VV\rangle\,\rangle = \frac{1}{\sqrt{2}}(a|H\rangle + b|V\rangle)$$

and thus

$$\mathrm{pr}_1 = \frac{1}{2}(a^*\langle H| + b^*\langle V|)(a|H\rangle + b|V\rangle) = \frac{1}{2}|a|^2 + |b|^2 = \frac{1}{2}.$$

The argument for the second element of Alice's basis and for the other Bell states is similar.

Solution to Exercise 2.20. Let us first analyze the second scenario. Alice's measurement will generate the state

$$\hat{P}_{i,\mathrm{Alice}} |\Psi\rangle = |v_i\rangle_{\mathrm{Alice}} \otimes \langle v_i| \, \Psi\rangle_{\mathrm{Bob}}\,,$$

where i is random and the probability is incorporated into the norm of the above state. If Bob now performs his measurement, we obtain

$$\hat{P}_{j,\mathrm{Bob}} \left(\hat{P}_{i,\mathrm{Alice}} |\Psi\rangle\right) = |v_i\rangle_{\mathrm{Alice}} \otimes |w_j\rangle_{\mathrm{Bob}} \langle w_j| \left(\langle v_i| \, \Psi\rangle\right). \tag{7.10}$$

But, as we found in Ex. 2.18, $\langle w_j| (\langle v_i| \, \Psi\rangle) = \langle v_i\, w_j| \, \Psi\rangle$. Accordingly, the state (7.10) is equal to the state

$$\hat{P}_{ij} |\Psi\rangle = |v_i\rangle_{\mathrm{Alice}} \otimes |w_j\rangle_{\mathrm{Bob}} \langle v_i\, w_j| \, \Psi\rangle \tag{7.11}$$

that is produced after the measurement in the first scenario. Accordingly, the probabilities associated with these states are also equal.

The equivalence of the third scenario to the first one is proven similarly.

Solution to Exercise 2.21. If Alice detects $|\theta\rangle$, Bob's photon will project onto

$$\begin{aligned}\left\langle\theta_{\text{Alice}}\right|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(\langle\theta|H\rangle|V\rangle_{\text{Bob}}-\langle\theta|V\rangle|H\rangle_{\text{Bob}})\\ &= \frac{1}{\sqrt{2}}(\cos\theta|V\rangle_{\text{Bob}}-\sin\theta|H\rangle_{\text{Bob}})=\frac{i}{\sqrt{2}}\left|\frac{\pi}{2}+\theta\right\rangle_{\text{Bob}},\end{aligned}\tag{7.12}$$

i.e. the linear polarization state oriented at angle $\pi/2+\theta$ to horizontal. The probability for Alice to detect $|\theta\rangle$ is $|1/\sqrt{2}|^2=1/2$. Similarly, if Alice's result is $|\pi/2+\theta\rangle$, Bob's state becomes $|\theta\rangle$ (up to a phase factor). The probability of this outcome is also $1/2$.

The result of this exercise is generalized in Ex. 2.24. Also note that the fact that both probabilities equal $1/2$ can also be obtained from Ex. LocMesBellEx.

Solution to Exercise 2.22. The result $\text{pr}_{\text{Bob},H}=\text{pr}_{\text{Bob},V}=1/2$ follows immediately from Exercises 2.19 and 2.20, but let us also calculate it explicitly. If Alice detects $|\theta\rangle$, Bob, who measures in the canonical basis, will observe H with probability $\cos^2\theta$ and V with probability $\sin^2\theta$. Taking into account the fact that Alice's observation occurs with probability $1/2$, we can write

$$\begin{aligned}\text{pr}_{\text{Alice},\theta;\text{Bob},H} &= \cos^2\theta/2;\\ \text{pr}_{\text{Alice},\theta;\text{Bob},V} &= \sin^2\theta/2.\end{aligned}$$

Similarly, the probabilities associated with Alice detecting $|\pi/2+\theta\rangle$ are

$$\begin{aligned}\text{pr}_{\text{Alice},\pi/2+\theta;\text{Bob},H} &= \sin^2\theta/2;\\ \text{pr}_{\text{Alice},\pi/2+\theta;\text{Bob},V} &= \cos^2\theta/2.\end{aligned}$$

Now we can find the probability for bob to obtain each result:

$$\begin{aligned}\text{pr}_{\text{Bob},H}=\text{pr}_{\text{Alice},\theta;\text{Bob},H}+\text{pr}_{\text{Alice},\pi/2+\theta;\text{Bob},H}=1/2;\\ \text{pr}_{\text{Bob},V}=\text{pr}_{\text{Alice},\theta;\text{Bob},V}+\text{pr}_{\text{Alice},\pi/2+\theta;\text{Bob},V}=1/2.\end{aligned}$$

Solution to Exercise 2.23. Acting similarly to Ex. 2.21, we find

$$\left\langle R_{\text{Alice}}\right|\Psi^-\rangle=\frac{1}{\sqrt{2}}(\langle R|H\rangle|V\rangle_{\text{Bob}}-\langle R|V\rangle|H\rangle_{\text{Bob}})=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|V\rangle_{\text{Bob}}-\frac{-i}{\sqrt{2}}|H\rangle_{\text{Bob}}\right)=\frac{i}{\sqrt{2}}|L\rangle_{\text{Bob}},$$

the left circular polarized state. Similarly, if Alice detects $|L\rangle$, the state remotely prepared with Bob will be R. The probabilities of both events are $1/2$.

Solution to Exercise 2.24. Using $|\theta\rangle=\cos\theta|H\rangle+\sin\theta|V\rangle$, we find

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left[\left|\theta_A(\frac{\pi}{2}+\theta)_B\right\rangle-\left|(\frac{\pi}{2}+\theta)_A\theta_B\right\rangle\right]\\ &=\frac{1}{\sqrt{2}}\left[(\cos\theta|H\rangle+\sin\theta|V\rangle)\otimes(-\sin\theta|H\rangle+\cos\theta|V\rangle)\right.\\ &\quad\left.-(-\sin\theta|H\rangle+\cos\theta|V\rangle)\otimes(\cos\theta|H\rangle+\sin\theta|V\rangle)\right]\\ &=\frac{1}{\sqrt{2}}\left[(\cos^2\theta+\sin^2\theta)|HV\rangle-(\cos^2\theta+\sin^2\theta)|VH\rangle=\left|\Psi^-\right\rangle\right].\end{aligned}$$

Solution to Exercise 2.25. By analogy to Ex. 2.24 we notice that the state $|\Psi^-\rangle$ can be expressed as

$$\left|\Psi^-\right\rangle=\frac{1}{\sqrt{2}}\left(\left|\tilde{H}\tilde{V}\right\rangle-\left|\tilde{V}\tilde{H}\right\rangle\right),\tag{7.13}$$

where the states $\left|\tilde{H}\right\rangle=\alpha|H\rangle+\beta|V\rangle$ and $\left|\tilde{V}\right\rangle=-\beta^*|H\rangle+\alpha^*|V\rangle$ form an orthonormal basis and $\left|\tilde{H}\right\rangle$ is the state Alice desires to prepare at Bob's location. From Eq. (7.13) we find that Alice should measure in the basis $\{\left|\tilde{H}\right\rangle,\left|\tilde{V}\right\rangle\}$. The desired measurement result for Alice is $\left|\tilde{V}\right\rangle$, it occurs with a probability of $1/2$.

Solution to Exercise 2.26. Let us assume that Alice has measured her photon in the canonical basis. The resulting states for Bob are a)

$$\langle H_{\text{Alice}}|\ \Psi\rangle = |H\rangle/\sqrt{5};$$
$$\langle V_{\text{Alice}}|\ \Psi\rangle = 2|V\rangle/\sqrt{5}.$$

Accordingly, the verbal description of Bob's photon is "either $|H\rangle$ with probability 1/5 or $|H\rangle$ with probability 4/5".

b)

$$\langle H_{\text{Alice}}|\ \Psi\rangle = (|H\rangle+|V\rangle)/\sqrt{3} = \sqrt{2/3}\,|+\rangle\,;$$
$$\langle V_{\text{Alice}}|\ \Psi\rangle = |V\rangle/\sqrt{3}.$$

This state is verbally described as "either $|+\rangle$ with probability 2/3 or $|V\rangle$ with probability 1/3". Note that, because two of the terms in the initial state $|\Psi\rangle$ have Alice's photon horizontally polarized, Alice's measurement does not destroy the coherence between them. This can also be seen by rewriting the initial state as

$$|\Psi\rangle = \sqrt{2/3}\,|H+\rangle + \sqrt{1/3}\,|VV\rangle\,.$$

Solution to Exercise 2.27. The number of terms that contain k out of n results with horizontal polarization is given by the combinatoric expression

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!}.$$

Since the total number of the superposition terms equals 2^n, the fraction of the terms we are interested in equals

$$\left(\begin{array}{c} n \\ k \end{array}\right)\Big/2^n = \frac{n!}{2^n k!(n-k)!} \tag{7.14}$$

In the framework of Copenhagen interpretation, the probability of either result in each measurement is 1/2. The probability of each specific sequence of results for n measurements equals $(1/2)^n$. The number of possible sequences that contain k results with horizontal polarization equals $\left(\begin{array}{c} n \\ k \end{array}\right)$. Accordingly, the probability to obtain any one of these sequences is given by Eq. (7.14).

Solution to Exercise 2.28. One can model probabilistic behavior in Alice's and Bob's apparata by random number generators such that each of the numbers M_A, N_A, M_B and N_B are deterministic functions of the state of the particle received from the sender and the value received from the random number generator. The rest of the argument remains the same.

Solution to Exercise 2.29. We first express the operators $\hat{M}_A$, $\hat{N}_A$, $\hat{M}_B$ and $\hat{N}_B$ (where θ is 0, $\pi/8$, $\pi/4$, or $3\pi/8$) in the canonical basis. Because all these operators can be written in the form $|\theta\rangle\langle\theta| - \left|\frac{\pi}{2}+\theta\right\rangle\left\langle\frac{\pi}{2}+\theta\right|$, we find

$$\begin{aligned}
&|\theta\rangle\langle\theta| - \left|\frac{\pi}{2}+\theta\right\rangle\left\langle\frac{\pi}{2}+\theta\right| \\
&= (\cos\theta\,|H\rangle + \sin\theta\,|V\rangle)(\cos\theta\,\langle H| + \sin\theta\,\langle V|) - (-\sin\theta\,|H\rangle + \cos\theta\,|V\rangle)(-\sin\theta\,\langle H| + \cos\theta\,\langle V|) \\
&= (\cos^2\theta - \sin^2\theta)\,|H\rangle\langle H| - (\cos^2\theta - \sin^2\theta)\,|V\rangle\langle V|\,2\cos\theta\sin\theta\,|H\rangle\langle V| + 2\cos\theta\sin\theta\,|V\rangle\langle H| \\
&= \cos(2\theta)(|H\rangle\langle H| - |V\rangle\langle V|) + \sin(2\theta)(|H\rangle\langle V| + |V\rangle\langle H|),
\end{aligned} \tag{7.15}$$

and thus

$$\begin{aligned}
\hat{M}_A &= |H\rangle\langle H| - |V\rangle\langle V| = \hat{\sigma}_z; \\
\hat{M}_B &= \frac{1}{\sqrt{2}}(|H\rangle\langle H| - |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|); \\
\hat{N}_A &= |H\rangle\langle V| + |V\rangle\langle H| = \hat{\sigma}_x; \\
\hat{N}_B &= \frac{1}{\sqrt{2}}(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|).
\end{aligned} \tag{7.16}$$

Next, we find the desired expectation values.

a) To determine $\langle\Psi^-|\,\hat{M}_A\otimes\hat{M}_B\,|\Psi^-\rangle$, we first calculate $\hat{M}_A\,|\Psi^-\rangle$ (because $\hat{M}_A$ and $\hat{M}_B$ live in different linear spaces, they commute, so we can apply them in any order). The operator $\hat{M}_A$ acts on Alice's photon, leaving the horizontal polarization unchanged, but multiplying the vertical polarization state by -1:

$$\hat{M}_A\,|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|H\rangle\langle H| - |V\rangle\langle V|)_A(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle).$$

Next, we act with the operator $\hat{M}_B$ on bob's photon:

$$\begin{aligned}\hat{M}_A\otimes\hat{M}_B\,|\Psi^-\rangle &= \frac{1}{2}(|H\rangle\langle H| - |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle + |VH\rangle)\\ &= \frac{1}{2}(|VH\rangle - |HV\rangle + |HH\rangle + |VV\rangle)\end{aligned}$$

and finally

$$\langle\Psi^-|\,\hat{M}_A\otimes\hat{M}_B\,|\Psi^-\rangle = \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(|VH\rangle - |HV\rangle + |HH\rangle + |VV\rangle) = -\frac{1}{2\sqrt{2}}.$$

Of course, the same calculation could have been carried out in the matrix form, akin to Ex. 2.13.

b) The second matrix element is found in a similar manner:

$$\begin{aligned}&\langle\Psi^-|\,\hat{M}_A\otimes\hat{N}_B\,|\Psi^-\rangle\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(|H\rangle\langle H| - |V\rangle\langle V|)_A(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle - |VH\rangle)\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(-|H\rangle\langle H| + |V\rangle\langle V| + |H\rangle\langle V| + |V\rangle\langle H|)_B(|HV\rangle + |VH\rangle)\\ &= \frac{1}{2\sqrt{2}}(\langle HV| - \langle VH|)(-|VH\rangle + |HV\rangle + |HH\rangle + |VV\rangle) = \frac{1}{2\sqrt{2}}.\end{aligned}$$

c) The third and fourth matrix elements could also be found by a direct calculation. The calculation can however be avoided if we remember that the state $|\Psi^-\rangle$ is isotropic. If both Alice and Bob rotate their reference frames by an angle $\pi/8$, state $|\Psi^-\rangle$ will remain unchanged, operator $\hat{N}_A$ in Alice's space will become $\hat{M}_B$, and operator $\hat{M}_B$ in Bob's space will become $\hat{M}_A$. In the new reference frame, we thus need to calculate the expectation value of the operator $\hat{M}_B\otimes\hat{M}_A$. Because state $|\Psi^-\rangle$ is antisymmetric with respect to switching Alice and Bob, the desired expectation value equals that of $\hat{M}_A\otimes\hat{M}_B$ determined in part (a), i.e. $-1/2\sqrt{2}$.

d) If we rotate Alice's and Bob's reference frames by $\pi/4$, operators $\hat{N}_A$ and $\hat{N}_B$ will become $\hat{M}_A$ and $\hat{M}_B$, respectively. The desired expectation value is once again equal to $\langle\hat{M}_A\otimes\hat{M}_B\rangle = -1/2\sqrt{2}$

Solution to Exercise 2.30. When one of the parties presses a button, a local measurement on $|\Psi^-\rangle$ in the basis $\{|\theta\rangle, |\pi/2+\theta\rangle\}$ (where θ is 0, $\pi/8$, $\pi/4$, or $3\pi/8$) is performed. As was found in Ex. 2.23, such a measurement will return either results with a probability 1/2, which means that the values $+1$ and -1 will be displayed with equal probabilities.

Solution to Exercise 2.31. Left for the reader as an independent exercise.

Solution to Exercise 2.32. Suppose cloning were possible. Subjecting two orthogonal states $|a_1\rangle$ and $|a_2\rangle$ and their linear superposition to cloning, we would obtain

$$|a_1\rangle\otimes|0\rangle \to |a_1\rangle\otimes|a_1\rangle \tag{7.17}$$
$$|a_2\rangle\otimes|0\rangle \to |a_2\rangle\otimes|a_2\rangle \tag{7.18}$$
$$\frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}\otimes|0\rangle \to \frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}\otimes\frac{|a_1\rangle+|a_2\rangle}{\sqrt{2}}. \tag{7.19}$$

But every quantum operation, including cloning, must be linear. Using linearity, adding Eqs. (7.17) and (7.18) together, we find

$$\frac{|a_1\rangle + |a_2\rangle}{\sqrt{2}} \otimes |0\rangle = \frac{|a_1\rangle \otimes |0\rangle + |a_2\rangle \otimes |0\rangle}{\sqrt{2}} \to \frac{|a_1\rangle \otimes |a_1\rangle + |a_2\rangle \otimes |a_2\rangle}{\sqrt{2}},$$

which is in contradiction with Eq. (7.19).

Solution to Exercise 2.33. If cloning were possible, Alice and Bob could implement the following protocol. They start with sharing an entangled state, e.g. $|\Psi^-\rangle$. When Alice needs to send a message to Bob, she encodes this message in the value of angle θ between 0 and $\pi/2$, and then performs a measurement of her photon in the basis $\{|\theta\rangle, \left|\frac{\pi}{2}+\theta\right\rangle\}$, thus instantly remotely preparing one of these two states at Bob's station. Bob makes many copies of this state and performs quantum tomography (see Defn. 1.14) on them, thus determining the value of θ with an arbitrarily high precision. Then he decodes the value and reads Alice's message.

Solution to Exercise 2.34. N qubits is a 2^N-dimensional quantum system. No matter how Alice encodes information in this system, Bob can perform only one quantum measurement on it. His measurement will produce one of 2^N possible outcomes, so the total number of different messages that can be encoded in N qubits is 2^N. The capacity of N bits of classical information is exactly the same.

Solution to Exercise 2.35.

$$(\hat{\sigma}_z)_A \left|\Psi^-\right\rangle = (|H\rangle\langle H| - |V\rangle\langle V|)_A \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) = \left|\Psi^+\right\rangle;$$

$$(\hat{\sigma}_x)_A \left|\Psi^-\right\rangle = (|H\rangle\langle V| + |V\rangle\langle H|)_A \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(|VV\rangle - |HH\rangle) = -\left|\Phi^-\right\rangle;$$

$$(\hat{\sigma}_y)_A \left|\Psi^-\right\rangle = (-i\,|H\rangle\langle V| + i\,|V\rangle\langle H|)_A \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) = \frac{1}{\sqrt{2}}(i\,|HH\rangle + i\,|VV\rangle) = i\left|\Phi^+\right\rangle.$$

Solution to Exercise 2.36.

$$\begin{aligned} |\text{input}\rangle = |\chi\rangle \otimes \left|\Psi^-\right\rangle &= (\alpha\,|H\rangle + \beta\,|V\rangle) \otimes \frac{1}{\sqrt{2}}(|H\rangle \otimes |V\rangle - |V\rangle \otimes |H\rangle) \\ &= \frac{1}{\sqrt{2}}(\alpha\,|HHV\rangle - \alpha\,|HVH\rangle + \beta\,|VHV\rangle - \beta\,|VVH\rangle). \end{aligned} \tag{7.20}$$

Solution to Exercise 2.37. From the definition of Bell states, we find

$$|HH\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}}; \tag{7.21}$$

$$|HV\rangle = \frac{|\Psi^+\rangle + |\Psi^-\rangle}{\sqrt{2}}; \tag{7.22}$$

$$|VH\rangle = \frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}; \tag{7.23}$$

$$|VV\rangle = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}}. \tag{7.24}$$

Solution to Exercise 2.38. Using the results of the two previous exercises, we find

$$\begin{aligned} |\text{input}\rangle &= \frac{1}{2}\left(\alpha\left|\Phi^+V\right\rangle + \alpha\left|\Phi^-V\right\rangle - \alpha\left|\Psi^+H\right\rangle - \alpha\left|\Psi^-H\right\rangle\right. \\ &+ \left.\beta\left|\Psi^+V\right\rangle - \beta\left|\Psi^-V\right\rangle - \beta\left|\Phi^+H\right\rangle + \beta\left|\Phi^-H\right\rangle\right). \end{aligned} \tag{7.25}$$

Solution to Exercise 2.39. Factoring out the Bell states in Eq. (7.25), we obtain

$$\begin{aligned} |\text{input}\rangle &= \frac{1}{2}\left|\Psi^-\right\rangle(-\alpha\,|H\rangle - \beta\,|V\rangle) + \frac{1}{2}\left|\Psi^+\right\rangle(-\alpha\,|H\rangle + \beta\,|H\rangle) \\ &+ \frac{1}{2}\left|\Phi^-\right\rangle(\alpha\,|V\rangle + \beta\,|H\rangle) + \frac{1}{2}\left|\Phi^+\right\rangle(\alpha\,|V\rangle - \beta\,|H\rangle). \end{aligned} \tag{7.26}$$

A measurement by Alice will randomly select one of the four terms in the above equation and prepare the corresponding state at Bob's station (see Ex. 2.40). Because the norm of each term is 1/2, the probability of each result is $(1/2)^2 = 1/4$. Note that these probabilities do not depend on the parameters α and β of the source state, so by performing the measurement in the Bell basis, Alice does not learn anything about this state.

Solution to Exercise 2.40.

- If Alice detects $|\Psi^-\rangle$, Bob's photon will project onto $-(\alpha|H\rangle + \beta|V\rangle)$, which, up to an overall phase factor, is identical to the source state $|\chi\rangle$. In this case Bob does not need to do anything.
- If Alice detects $|\Psi^+\rangle$, Bob's photon will project onto $-(\alpha|H\rangle - \beta|V\rangle)$. To obtain $|\chi\rangle$, Bob will need to perform an operation which does not change the horizontally polarized photon, but applies a phase factor of (-1) to the vertically polarized. This operation is achieved by Pauli operator $\hat{\sigma}_z = |H\rangle\langle H| - |V\rangle\langle V|$.
- If Alice detects $|\Phi^-\rangle$, Bob's photon will project onto $(\beta|H\rangle + \alpha|V\rangle)$. In this case, Bob needs to flip between horizontal and vertical polarizations, which is done by Pauli operator $\hat{\sigma}_x = |H\rangle\langle V| + |V\rangle\langle H|$.
- If Alice detects $|\Phi^+\rangle$, Bob's photon will project onto $(-\beta|H\rangle + \alpha|V\rangle)$. Bob must *both* flip the polarizations and shift the phase of one of the polarizations, i.e. apply $\hat{\sigma}_z\hat{\sigma}_x = |H\rangle\langle V| - |V\rangle\langle H|$. Note that we can write this operator as $-i\hat{\sigma}_y$.

By receiving classical communication from Alice concerning which Bell state she has detected and performing one of the Pauli operations on his photon, Bob receives a copy of the source state $|\chi\rangle$. The original source state is destroyed in Alice's measurement, so the no-cloning theorem is not violated.

Solution to Exercise 2.41. The answer is positive in all three cases. The details are left for the reader as an independent exercise.

Solution to Exercise 2.42. Proceeding in a similar fashion to the quantum teleportation argument, we find:

$$
\begin{aligned}
\left|\Psi^-\Psi^-\right\rangle_{ABCD} &= \frac{1}{2}(|HVHV\rangle - |HVVH\rangle - |VHHV\rangle + |VHVH\rangle)_{ABCD} \\
&= \frac{1}{2\sqrt{2}}\left[|H\rangle\left(\left|\Psi^+\right\rangle - \left|\Psi^-\right\rangle\right)|V\rangle - |H\rangle\left(\left|\Phi^+\right\rangle - \left|\Phi^-\right\rangle\right)|H\rangle\right. \\
&- \left.|V\rangle\left(\left|\Phi^+\right\rangle + \left|\Phi^-\right\rangle\right)|V\rangle + |V\rangle\left(\left|\Psi^+\right\rangle + \left|\Psi^-\right\rangle\right)|H\rangle\right]_{ABCD} \\
&= \frac{1}{2\sqrt{2}}\left[\left|\Psi^-\right\rangle_{BC}(-|HV\rangle + |VH\rangle)_{AD} + \left|\Psi^+\right\rangle_{BC}(|HV\rangle + |VH\rangle)_{AD}\right. \\
&+ \left.\left|\Phi^-\right\rangle_{BC}(|HH\rangle - |VV\rangle)_{AD} + \left|\Phi^+\right\rangle_{BC}(-|HH\rangle - |VV\rangle)_{AD}\right] \\
&= \frac{1}{2}\left[-\left|\Psi^-\right\rangle_{BC}\left|\Psi^-\right\rangle_{AD} + \left|\Psi^+\right\rangle_{BC}\left|\Psi^+\right\rangle_{AD} + \left|\Phi^-\right\rangle_{BC}\left|\Phi^-\right\rangle_{AD} - \left|\Phi^+\right\rangle_{BC}\left|\Phi^+\right\rangle_{AD}\right].
\end{aligned}
$$

Detecting photons B and C in a particular Bell state will entangle the remaining two photons, projecting them onto the same Bell state. As in the case of quantum teleportation, the probability of each measurement outcome is 1/4.